ACTA SCIENTIAE
ISSN: 2178-7727
DOI: 10.17648/acta.scientiae.5050

Researches about curves sketching on the perspective of global interpretation of figural units focusing on high school

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Received 14 February. 2019. Accepted, after revision, on 31 Oct. 2019
Designated editor: Claudia Lisete Oliveira Groenwald

ABSTRACT

A teaching approach based on the global interpretation of figurative properties prioritizes the curve sketching and understanding from the conversions among algebraic, graphical and linguistic registers, more precisely the identification of basic units (graphical, symbolic and linguistic) and the verification of how they related with each other. Raymond Duval presents this approach in an outline work of a related linear function using a resource for the global interpretation of the parameters of the algebraic expression: $y = ax + b$, emphasizing the relationship between these parameters and the graphical visual variables: inclination direction, tracing angles with the axes, and tracing position relatively to the vertical axis origin. Other authors have proposed works under this perspective with a focus on High School. Among them: Moretti (2003), for the quadratic function; Silva (2008), for the exponential, logarithmic and trigonometric functions; Menoncini & Moretti (2017), for the modular function; Martins (2017), for curves whose expressions are in parametric form; Moretti, Ferraz & Ferreira (2008), for more complex functions of college teaching; and Pasa (2017), for polynomial functions of the second and third degree. In this article, we present these works, complementing Pasa & Moretti (2016), presenting the resources used in each of them that allow changing verifications which the graph changes generates in the algebraic expression and vice versa and the identification of the visual variables and units related to the modifications.

Keywords: Curve outline; Global interpretation of figurative properties; High school.

Pesquisas sobre o esboço de curvas na perspectiva da interpretação global das unidades figurais com foco no ensino médio

RESUMO

Uma abordagem de ensino baseada na interpretação global de propriedades figurais prioriza o esboço e a compreensão de uma curva a partir das conversões entre registros algébrico, gráfico e linguístico, mais precisamente da identificação de unidades básicas (gráficas, simbólicas e linguísticas) e da verificação de como estas se relacionam. Raymond Duval apresenta esta

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abordagem em um trabalho de esboço da reta da função afim utilizando como recurso para a interpretação global os parâmetros da expressão algébrica $y = ax + b$, enfatizando a relação entre estes parâmetros e as variáveis visuais gráficas: sentido da inclinação, ângulos do traçado com os eixos e a posição do traçado em relação à origem do eixo vertical. Outros autores propuseram trabalhos nesta perspectiva com foco no ensino médio, entre eles Moretti (2003), para a função quadrática; Silva (2008), para as funções exponencial, logarítmica e trigonométrica; Menoncini e Moretti (2017), para a função modular; Martins (2017), para curvas cujas expressões estão na forma paramétrica; Moretti, Ferraz e Ferreira (2008), para funções mais complexas do ensino universitário e Pasa (2017), para funções polinomiais do segundo e terceiro grau. Neste artigo, expomos estes trabalhos, complementando Pasa e Moretti (2016), apresentando os recursos utilizados em cada um deles que possibilitam a verificação das modificações que a mudança do gráfico gera na expressão algébrica e vice e versa e a identificação das variáveis visuais e unidades simbólicas pertinentes relacionadas às modificações.

Palavras-chave: Esboço de curvas; Interpretação global de propriedades figurais; Ensino médio.

INTRODUCTION

In various areas of knowledge, curves sketching and its understanding are essential activities for interpreting graphs and phenomena. However, these activities, which are widely used at all levels of education, are accompanied by numerous difficulties related to teaching and learning, which prompt researches on the reasons for these difficulties and pedagogical possibilities that addresses these issues.

From the perspective of Raymond Duval's theory of registers of semiotics representations, the whole analysis of the acquisition and construction of mathematical knowledge permeates three closely linked phenomena: the diversification on semiotics representations registers, the differentiation between representant and represented, or between form and content of a semiotic representation, and the coordination (conversion) between the different registers of semiotic representation. These aspects, linked to non-congruence issues[1] among the registers are, for this author, the sources of the difficulties in understanding mathematics.

In the case of curve sketching activity, Duval (2011)[2] states that the problems lie in the transition between graphic and algebraic register, more specifically, in the lack of knowledge of the semiotic correspondence rules between these representations registers. In teaching, the approach often used in working with curve sketching prioritizes and highlights activities of moving from algebraic to graphic expression through a "point to point" approach, which entails few difficulties. However, in the opposite direction, that is, when moving from graph to algebraic expression, students have numerous difficulties due to the fact that significant units of a graph are not determined in relation to the points found but by visual values of the graph.

[1] The issues involving congruence and non-semantic congruence between semiotics representations registers can be further investigated in Duval (2012) and Duval (2004, p.49-61).

[2] Translation by Méricles Thadeu Moretti de Duval, R. *Graphiques e équations: l'articulation de registres*. *Annales de Didactique et de Sciences Cognitives*. Strasbourg: IREM, 1988.

Duval (2011) suggests working with curves sketching from a global interpretation approach to figurative properties. From the perspective of this approach, the effective understanding of a curve and what it represents, permeates the knowledge of various registers of semiotic representation of the outlining function/equation; the identification of visual variables, significant symbolic units and, plus, the verification of how changes in visual variables influences significant symbolic units and vice versa. The visual variables refer to representing registers graphically, while significant symbolic units refer to representing it algebraically.

From the perspective of this approach, several studies have been carried out to proposing varied possibilities of articulations between visual variables and significant symbolic units. In this article, we present works with curves sketching of functions for high school contexts, complementary to those exposed in Pasa & Moretti (2016). In each paper, it is highlighted the articulation resources used and its examples in order to clarify the approach and verify changes that one register influences in the other.

RESEARCH RELATED TO THE CURVES SKETCHING BASED ON THE GLOBAL INTERPRETATION OF FIGURATIVE UNITS APPROACH

1. First-degree polynomial function

In the article entitled *Gráficos e equações: a articulação de dois registros*[3], Duval (2011) exposes the global interpretation of the figurative properties approach in the specific case of the line sketching of the first degree polynomial function, using as a resource the related function coefficients: $y = ax + b$. The global interpretation occurs, in this case, through the knowledge of semiotic correspondence rules between the graphic representation register and the algebraic one.

From the perspective of this work, the analysis of the congruence between the algebraic and graphic registers goes through the discrimination of the significant units, proper to each register, and the implicit transformations required for its change. Graphically, Duval (2011) highlights the visual variables as: *the inclination direction* (can assume two values), *the angles between the line and the axes* (can assume three values) and *the line position related with the origin of the vertical axis* (can assume three values). Algebraically, significant units are the explicit and implicit symbols, as can be seen in Table 1.

[3] N.T.: the article's title would be translated as, *Graphs and equations: the articulation of two registers*.

Table 1

Values and visual variables for $y = ax + b$ *in the cartesian plane.* (Duval, 2011, p.101)

Visual Variables	Values	Corresponding symbolic units	
Tilt Direction	Ascending	Coefficient> 0	No Signal
	Downward	Coefficient <0	Signal Presence -
Angles with the axés	Symmetric partition	Coeff. Variable = 1	Coef. unwritten
	Smaller angle	Coeff. Variable <1	Coef. Written
	Larger angle	Coeff. Variable> 1	Coef. Written
Axis position	Cut up	Adds constant	+ Sign
	Cut below	Subtracts constant	Signal -
	Cut at origin	No additive correction	No Signal

The values of the graphics visual variables, shown in the second column of Table 1, correspond to a significant unit in the straight-line algebraic expression (corresponding symbolic unit). From the understanding rules of correspondence between the significant units of each record, various analyzes of the function can be made. For example, given the function $y = 3x - 4$, with $a = 3$ and $b = -4$. According to Table 1, the inclination direction of this line is increasing since the coefficient, denoted as a, is greater than zero ($a > 0$) (presence of the signal +). As $a > 1$, the angle with x axis is greater and the subtracted constant ($b = -4$) informs that the line intersects the y axis below the x axis.

2. Second-degree polynomial function

Moretti (2003) has shown one way to make explicit the relationship between the visual variable representation and significant unit of algebraic writing of the quadratic function, in which the author uses *translation* as a resource. Commonly, in working with high school, the quadratic function is generally viewed as $y = ax^2 + bx + c$, with $a \neq 0$, b and c real constants, but this algebraic form has no congruence with graphical visual variables, which can lead to difficulties in the graph sketching.

In face of this, Moretti (2003) suggests parabolas sketching from a base parabola positioned vertically at the origin of the Cartesian plane, focusing on (0 p/2) and straight line rule $y = -p/2$, with $p \neq 0$ and p being the distance between the focus and the straight line. This base parable has the equation ($y = 1/2p$) x^2 which, when a⍰ is related to the equation $y = \frac{1}{2p}x^2$, we have $y = ax^2$, which makes possible to realize that the a⍰ sign depends on the p sign, that is, the a coefficient informs about the concavity, up or down. Parables of this kind, with a vertex in their origin, enable the student to more easily recognize the relationships between meaningful units of expression and graphic symbolic units.

As an example, let's look at the parabolic algebraic expression $y = 2x^2 - 4x - 6$ (expression (1)). In order to compare it to the function with vertex at the origin $y = 2x^2$, it is required a square complement treatment and rewrite it as $y\,(-8) = 2(x - (+1))^2$ (expression (2)). Thus, it can be concluded that the function curve $y = 2x^2 - 4x - 6$ can be obtained by two translation movements of $y = 2x^2$: *horizontal right side in 1 unit* and *vertical down in 8 units*. The vertex then passes from the origin (0,0) to (1,0) and then to (1, -8). This same procedure is done to obtain focus and straight line. The parabola sketching $y = 2x^2$ and $y = 2x^2 - 4x - 6$ are presented in Figure 1, below.

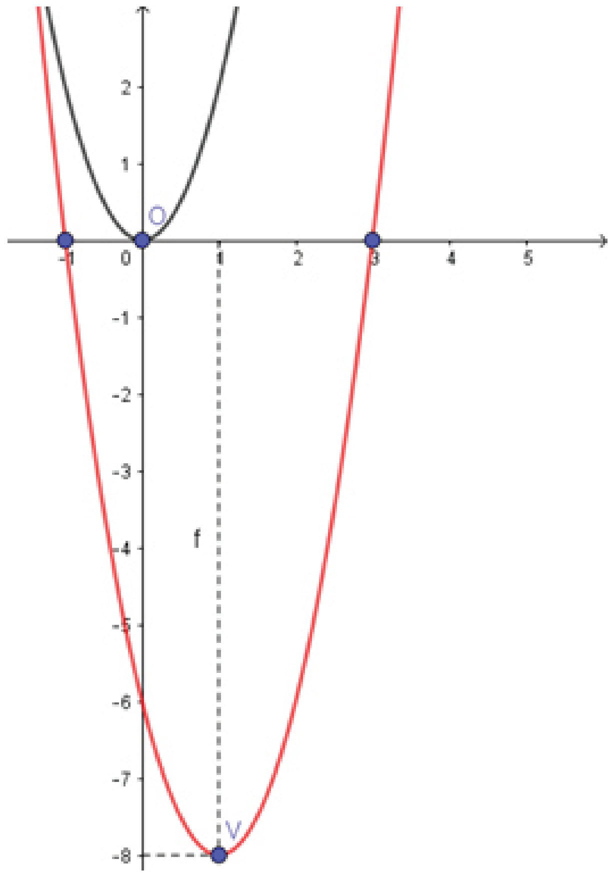


Figure 1. Parabola sketching $y = 2x^2 - 4x - 6$ from the parabola translation movements $y = 2x^2$.

According to Moretti (2003), expression (2) has a higher degree of semantic congruence with the translations described at the graphic level. This means that, for conversions between algebraic register and second-degree polynomial graphic functions, translational transformation can minimize non-congruence problems. Parabola sketching using translation can help the student perceive the set line/axis as an image representing an object described by an algebraic expression, realizing the implications of variations in the algebraic register, the graphic register and vice versa.

Moreover, this study with the quadratic function approximates the work with the parabola as a curve obtained by the geometric place of the equidistant points from a certain position called focus and a straight-line. Textbooks little relate these two moments in which the parabola is studied, which makes it difficult for students to relate and identify them as the same object.

Trigonometric, Exponential, and Logarithmic Functions

The curves sketching of the trigonometric, exponential and logarithmic functions are discussed in depth by Silva (2008) from visual variables such as *amplitude* and *period,* and features such as *translation* and *symmetry in parallel* with the significant units of algebraic expression. The author proposes using of the cognitive treatment operation to the algebraic and figural registers separately and parallel to a *base curve* in order to reach the figural and algebraic registers of the curve sketching.

In the case of the exponential function, in order to understand which changes in the coefficients of the curve's algebraic expression reflect changes in the graph, Silva (2008) considers a *base exponential curve* $y = a^x$, where $a \in \mathbb{R}$, $a > 1$ and thus concludes:

- the exponential $y = a^x$ where $a \in \mathbb{R}$, $0 < a < 1$: obtained by symmetry relative to the y axis from the exponential curve with inverse base;

- the exponential of the type $y = -a^x$ where $a \in \mathbb{R}_+^* - \{1\}$: obtained by symmetry with respect to the x axis from $y = a^x$;

- exponentials in the form $y = a^{x-(-d)}$: obtained by horizontal translation of d units to the left side of the base curve $y = a^x$.

- exponentials in the form $y = a^{x-(+d)}$: obtained by horizontal translation of d units to the right side of the base curve $y = a^x$.

- exponentials in the form $y - (+b) = a^x$: obtained by vertically translating b units upwards from the base curve $y = a^x$.

- exponentials in the form $y - (-b) = a^x$: obtained by vertical translation of b units below the base curve $y = a^x$.

- exponential in form $y - (\pm b) = a^{x-(\pm d)}$: obtained by vertical and horizontal translation of the base curve $y = a^x$, where the direction of displacement is given by the sign accompanying the values of b and d. Positive sign indicates upward displacement for b and to the right for d. On the other hand, the negative sign indicates downward displacement in case of b and to the left in case of d.

Take for example the curve sketching of the function $y = 2^{x-3} - 4$. We initially wrote $y - (-4) = 2^{x-(+3)}$ which can be obtained by vertical and horizontal translation of the base curve $y = 2^x$. In this case we have vertical translation of 4 units down and horizontal translation of 3 units to the right side. Figure 2, below, shows the curves.

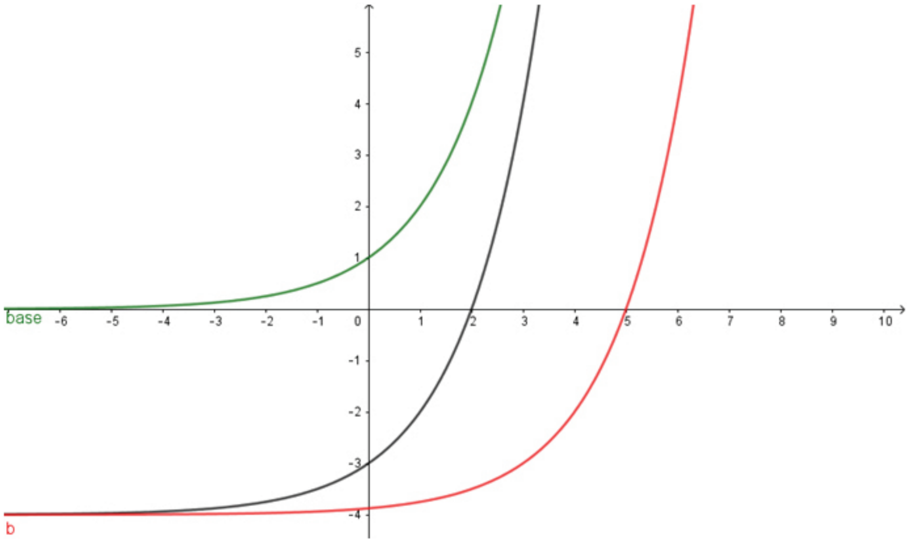


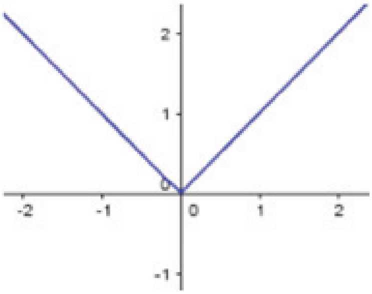
Figure 2. Curve sketching of the functions $y = 2^x$ (**base**), $y - (-4) = 2^x$ and $y - (-4) = 2^{x-(+3)}$.

Figure 2 makes possible to see the curve sketching (b) from the 4 vertical translation units downward and 3 horizontal translation units to the right side of the base curve, which can be obtained effortlessly via the point-to-point approach. This same analysis can be performed for trigonometric and logarithmic functions.

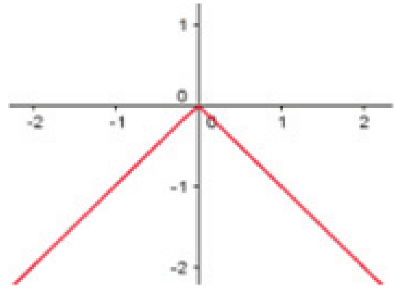
Linear Modular Function

Menoncini & Moretti (2017) investigate possibilities for working with linear modular functions from the perspective of global interpretation using the coefficients of algebraic representation as basic units. In this study, the authors performed three stages of analysis. The **first step** consists of visualizing curves sketching of linear modular functions and identifying the visual variables and their values, shown in Figures 3a and 3b below.

1.1 – A

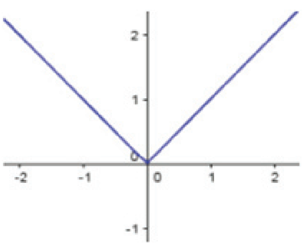


B



1.1 Visual variables	**Tracing angles** with axes
Visual Variable Values	A - The tracing is facing up B - The tracing is facing down C -

1.2 – A



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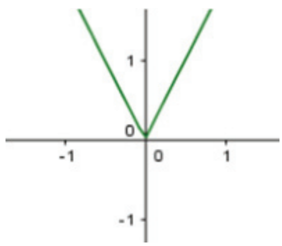
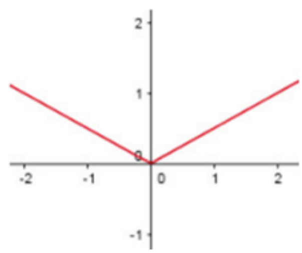


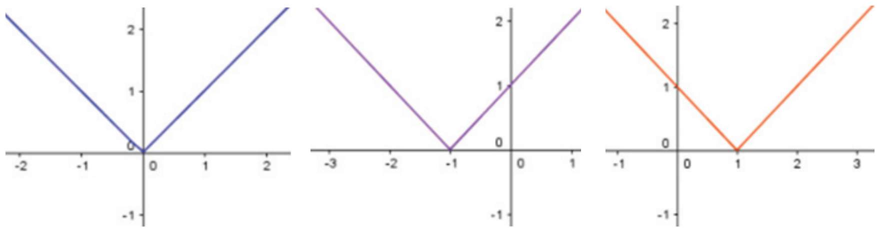
C



1.2 Visual Variables	**Tracing angles** with axes
Visual Variable Values	A- Symmetric Quadrant Division B- Angle with horizontal axis is greater than angle with vertical axis C- The angle with the horizontal axis is smaller than the angle with the vertical axis.

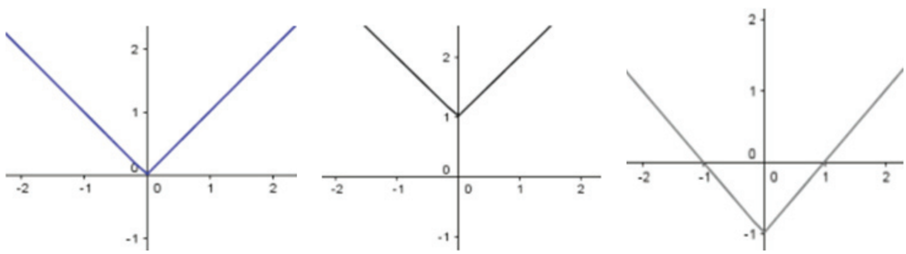
Figure 3a. Representative curves sketching of the linear modular function - Identification of values and visual variables for the tracing of the function in the Cartesian plane. (Menoncini, Moretti, 2017, pp. 129)

1.3 – A B C



1.3 Visuais variables	**Visual Variable Values**
Tracing **position** regarding the **horizontal axis** of the origin	A-The tracing passes through the **origin** B-The tracing moves to the **left side of the origin** C-The tracing moves to the **right side of the origin**

1.4 – A B C



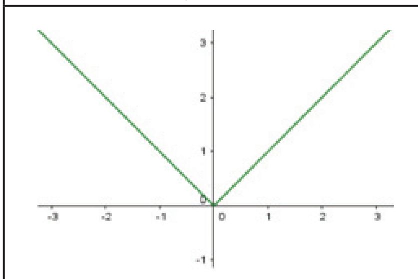
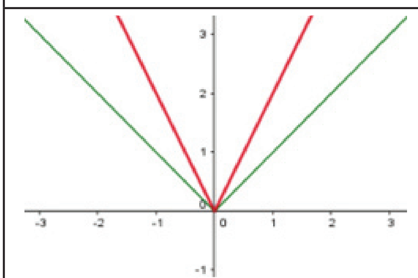
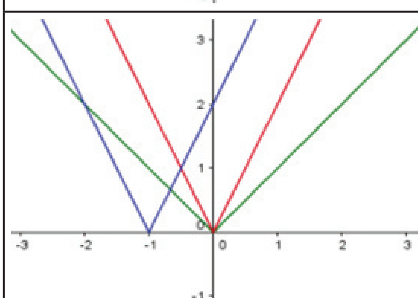
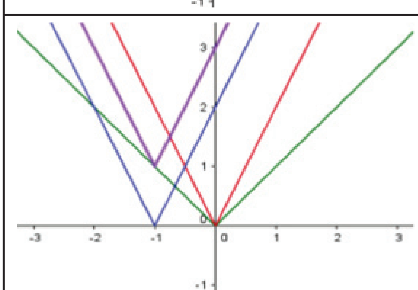
1.4 Visual variables	**Visual Variable Values**
Tracing **Position** regarding the **vertical axis** of the origin	A- The tracing passes through the **origin** B- The tracing moves **upward of the origin** C- The tracing moves **downward of the origin**

Figure 3b. Representative curves sketching of the linear modular function - Identification of values and visual variables for the tracing of the function in the Cartesian plane. (Menoncini, Moretti, 2017, pp. 129)

In the **second stage,** the authors associated the visual variables of Figures 3a and 3b to symbolic units considering that the algebraic expression of the absolute value in its canonical form is: $f(x) - (\pm a) = b\left|x - \frac{(\pm c)}{k}\right|$, with *a, b, c* and *k*, real constants. This association occurred from the base function of modular functions, defined as the real function $f(x) = |x| = \begin{cases} x, & se\ x \geq 0 \\ -x, & se\ x < 0 \end{cases}$, of $\mathbb{R}$ in $\mathbb{R}^+$. This function sketching is performed from the construction of semi-straight lines by assigning values to the variable and then by joining them, forming the trace.

Finally, in the **third step,** there was a general description of the curve characteristics and the establishment of correspondences between the algebraic significant units and the graphic visual units.

Using the same example used by the authors, the function $f(x) = 1 + 2\lvert x + 1\rvert$ whose canonical form is $f(x) - 1 = 2\lvert x - (-1)\rvert$ (Menoncini; Moretti, 2017, pp. 131) with coefficients $a = 1$, $b = 2$, $c = -1$ and $k = 1$ can be drawn from the base function $f(x) = \lvert x\rvert$, which gives the other functions as shown in Figure 4 below.

Algebraic Treatment of Function $f(x) = \lvert x\rvert$	Base curve sketching treatment
1 - Base Function $f(x) = \lvert x\rvert$	
2 - Assignment of the value 2 to the coefficient $f(x) = \lvert 2x\rvert = 2\lvert x\rvert$	
3 - Assignment of negative value 2 to constant c $f(x) = \lvert 2x - (-2)\rvert = 2\left\lvert x - \frac{-2}{2}\right\rvert$ $f(x) = 2\lvert x - (-1)\rvert$	
4- Assignment of positive value 1 to constant $f(x) - (1) = 2\lvert x - (-1)\rvert$	

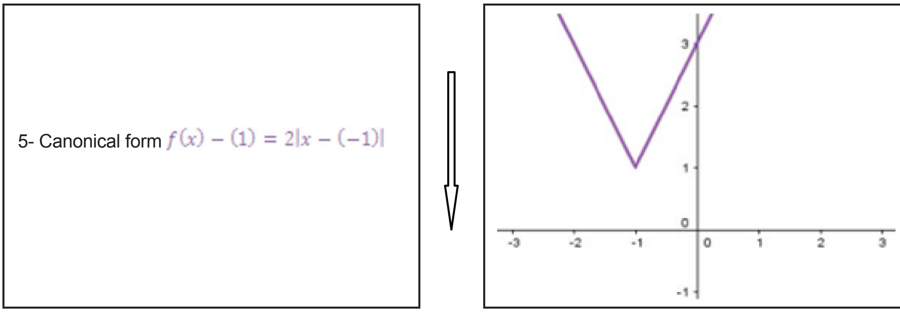


Figure 4. Curve tracing construction regarding to the function - $f(x) - 1 = 2|x - (-1)|$.[4]

Figure 4 shows the path taken to the curve sketching of the function $f(x) - 1 = 2|x - (-1)|$, from the base function $f(x) = |x|$. Thus, with $k = 2$, we have an increase in the curve sketching angle with the horizontal axis. Assigning $c = -2$, where $\frac{c}{k} = \frac{-2}{2} = -1$, the curve tracing is shifted one unit to the left. Next, one unit shifts upwards, because pois $a = +1$. Thus, at 5, we have the curve sketching after the algebraic modifications.

The correspondences between algebraic units and visual variables of linear modular functions of type $f(x) - (\pm a) = b|kx - (\pm c)|$ are:

- the coefficient b indicates the concavity of the stroke: facing up if $b > 0$ or downward if $b < 0$;

- the coefficient k indicates the opening tracing angle: symmetrical angle, if $k = 1$; angle with horizontal axis greater if $k > 1$ and angle with smaller horizontal axis if $k < 1$.

- constant term a indicates the tracing translation on the y axis: tracing moves a units upward if $a > 0$ or downward if $a < 0$;

- constant term c indicates the translation on the x axis: tracing moves $\frac{c}{k}$ units to the right if $\frac{c}{k} > 0$ or to the left if $\frac{c}{k} < 0$;

- the tracing vertex $V = (x_V, y_V)$ has coordinates $x_V = \frac{c}{k}$ and $y_V = \pm a$.

Thus, it is possible to make two-way associations between algebraic expression and visual variables, realizing that they complement each other and represent the same mathematical object.

Curves given by parametric equations

The straight-line and the parabola are studied by Martins (2017) as curves formed by the trajectory of a moving point in the plane, based on their parametric equations from

[4] Modified from Menoncini & Moretti (2017, pp. 132).

the perspective of the global interpretation of figurative units. For this study, the author uses algebraic manipulation in the parametric equations and a software that makes it possible to directly obtain the graphical representation of the curve and thus verify the figurative properties and their relationships with symbolic units.

Thus, given the straight-line that passes through the origin and point $P(x_0, y_0)$, exposed in Figure 5, the parametric equations are given by $r: \begin{cases} x = x_0.t \\ y = y_0.t \end{cases}$. In the case of any straight-line *s*, parallel to *r*, passing by the point $Q(x_1, y_1)$, shown in Figure 6, the parametric equations are given by $\begin{cases} x = x_1 + x_0.t \\ y = y_1 + y_0.t \end{cases}$.

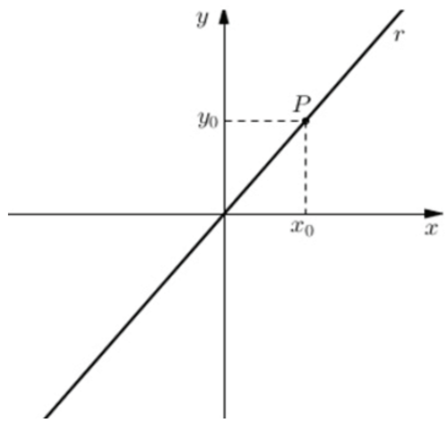


Figure 5. Straight-line *r* passes by the origin and by the point $P(x_0, y_0)$. (Martins, 2017, pp. 135)

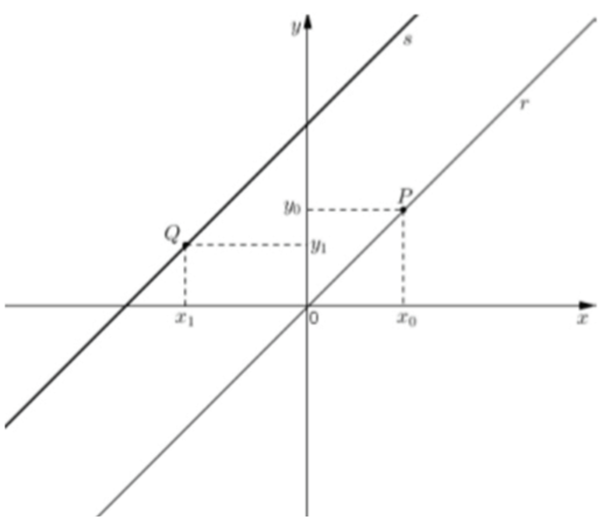


Figure 6. Straight-line *s*, parallel to *r*, passing through origin and point $Q(x_1, y_1)$. (Martins, 2017, pp. 136)

From simple algebraic manipulation we arrive at the equation of $s: y = \frac{y_0}{x_0}.x + \left(y_1 - \frac{y_0}{x_0}.x_1\right)$, (which is analyzed in Table 2 below.

Table 2
Visual variables and corresponding symbolic units for the straight-line given by parametric equations. (Martins, 2017, pp. 137)

Visual Variables	**Corresponding symbolic units**				**Values**
	x_0	y_0	x_0	y_0	Characteristics
Specific Cases	= 0		= 0		Straight-line coincides with the y axis
		= 0		= 0	Straight-line coincides with the x axis
		= 0		$\neq$ 0	Straight-line parallel to the x axis
	= 0		$\neq$ 0		Straight-line parallel to the y axis
			= 0		Straight-line passes through the origin
Tilt Direction	Sinais iguais				Ascending straight-line
	Sinais diferentes				Descending straight-line
Angle with the axes	$\lvert x_0 \rvert = \lvert y_0 \rvert$				Symmetric partition
	$\lvert x_0 \rvert < \lvert y_0 \rvert$				Angle greater (45º)
	$\lvert x_0 \rvert > \lvert y_0 \rvert$				Angle smaller (45º)
Axes positions	$y_1 - \frac{y_0}{x_0}.x_1 = 0$				Cut at origin (has no additive correction)
	$y_1 - \frac{y_0}{x_0}.x_1 > 0$				Cut upward (adds up $y_1 - \frac{y_0}{x_0}.x_1$)
	$y_1 - \frac{y_0}{x_0}.x_1 < 0$				Cut downward (subtract off $y_1 - \frac{y_0}{x_0}.x_1$)

According to Table 2, the line is analyzed from the algebraic expression generated by the parametric equations. The visual variables are the tilt direction, the angle with the axes and the axes position, while the corresponding symbolic units are observed from the coordinates of the points *P* and *Q* through which the line passes.

For the quadratic functions, Martins (2017) relates the visual variables to the corresponding symbolic units of the parametric equations of this function. In this case, Martins (2017) broadens the study by Moretti (2003), based on the analysis of the parametric writing coefficients. Thus, we initially analyzed vertex parables in the origin $\begin{cases} x = at^2 \\ y = bt \end{cases}$ (1), with horizontal symmetry axis and $\begin{cases} x = bt \\ y = at^2 \end{cases}$ (2), whose symmetry axis is vertical. The coefficients *a* and *b* directly influence the opening of the parabola. In order to analyze, for example, the opening of the parabola (1), one can take as reference the parabola given by the equations $\begin{cases} x = t^2 \\ y = t \end{cases}$, and from some sketches it can be concluded that the parabola will have larger opening when $0 < |a| < 1$ and a smaller opening if $|a| > 1$. On the other hand, being $|b| > 1$, the opening will be bigger and $0 < |b| <$ smaller. For functions of type $\begin{cases} x = at^2 + x_0 \\ y = bt + y_0 \end{cases}$ (3), with symmetry axis parallel to the *x* axis, or, $\begin{cases} x = bt + x_0 \\ y = at^2 + y_0 \end{cases}$ (4), with symmetry axis parallel to the *y* axis, Martins (2017) summarizes the analysis of the coefficients in Table 3 below.

Table 3

Relationships between some values, visual variables and significant symbolic units of the parable of parametric equations (3) and (4). (Martins, 2017, pp. 178)

2nd degree Term	Symmetry axis	Symbolic Units	Concavity	Vertex
t^2 in the x equation	Parallel to x axis	$a > 0$ $x_0 = y_0 = 0$	Right turned	(0, 0)
		$a > 0$ $x_0 \neq 0$ ou $y_0 \neq 0$		(x_0, y_0)
		$a < 0$ $x_0 = y_0 = 0$	Left turned	(0, 0)
		$a < 0$ $x_0 \neq 0$ ou $y_0 \neq 0$		(x_0, y_0)
t^2 in the y equation	Parallel to y axis	$a > 0$ $x_0 = y_0 = 0$	Upward turned	(0, 0)
		$a > 0$ $x_0 \neq 0$ ou $y_0 \neq 0$		(x_0, y_0)
		$a < 0$ $x_0 = y_0 = 0$	Downward turned	(0, 0)
		$a < 0$ $x_0 \neq 0$ ou $y_0 \neq 0$		(x_0, y_0)

Table 3 presents some parabola analyzes relating equations (3) and (4) coefficients to visual variables, axis of symmetry, concavity and vertex.

An example used by Martins (2017, pp. 181) is the parable of parametric equations $\begin{cases} x = -t^2 + 2 \\ y = -2t - 1 \end{cases}, t \in \mathbb{R}$. From a treatment it is possible to write $\begin{cases} x - (+2) = -t^2 \\ y - (-1) = -2t \end{cases}$ and, therefore, the reference for the sketching is the base curve $\begin{cases} x = -t^2 \\ y = -2t \end{cases}$, which has axis of symmetry in the x axis, concavity facing left because $a = -1 < 0$ and vertex (0, 0) because $x_0 = y_0 = 0$; shown in Figure 7.

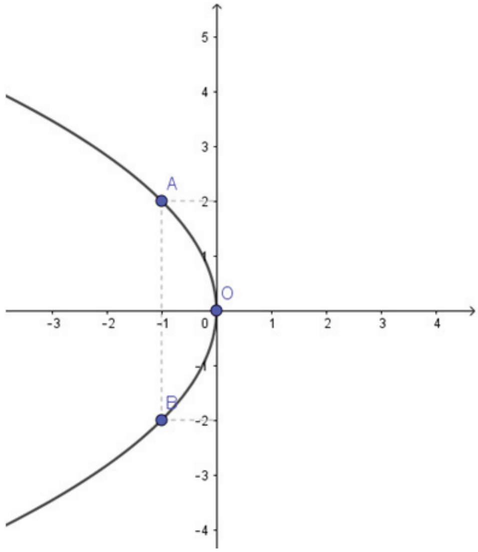


Figure 7. Curve sketching of Equation $\begin{cases} x = -t^2 \\ y = -2t \end{cases}, t \in \mathbb{R}$ based on table 3.

Based on Figure 7 and writing $\begin{cases} x - (+2) = -t^2 \\ y - (-1) = -2t \end{cases}$, it can be concluded that the parabola $\begin{cases} x = -t^2 + 2 \\ y = -2t - 1 \end{cases}, t \in \mathbb{R}$ has symmetry axis *x* axis, left-facing concavity and vertex $V(2, -1)$, in the base curve is moved horizontally 2 units to the right and vertically 1 unit down, as shown in Figure 8.

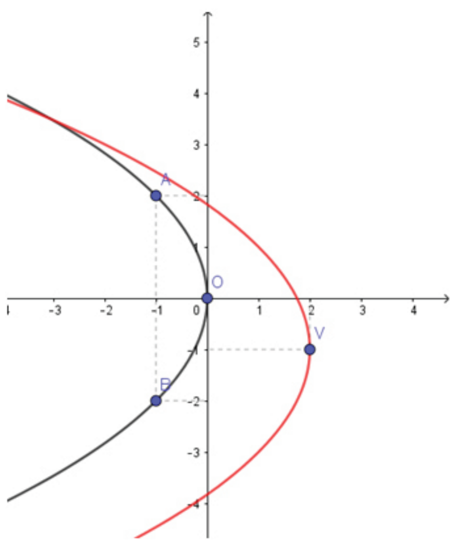
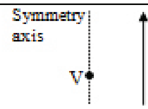
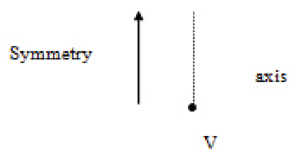
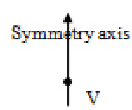
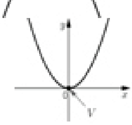
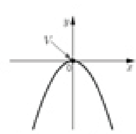


Figure 8. Curve sketching of the function $\begin{cases} x = -t^2 + 2 \\ y = -2t - 1 \end{cases}, t \in \mathbb{R}$.

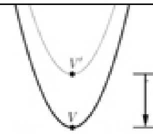
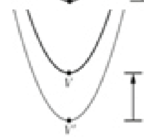
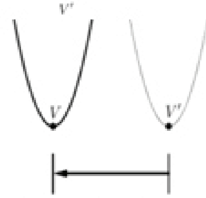
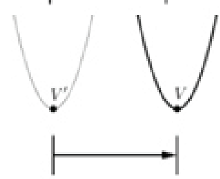
An interpretation of the parabolas of generic parametric equations $\begin{cases} x = bt + x_0 \\ y = at^2 + ct + y_0 \end{cases}$ (5) is presented by Martins (2017, pp. 190). In this article, we present only some relationships between visual variables and significant units as a way of exemplifying the author's reasoning for these parabolas. Moreover, the analysis of equations of type $\begin{cases} x = at^2 + ct + x_0 \\ y = bt + y_0 \end{cases}$ (6) is analogous.

Table 4

Some visual and symbolic characteristics of parabolas given by parametric equations of the type $\begin{cases} x = bt + x_0 \\ y = at^2 + ct + y_0 \end{cases}$*, with* $t \in \mathbb{R}$*,* $a \in \mathbb{R}^*$ *and* $b, c, x_0, y_0 \in \mathbb{R}$. (Adapted from Martins, 2017, pp. 190-195)

Meaningful linguistics units	Meaningful symbolic units	Visual Variables
2[nd] degree term in the y coordinate equation Symmetry axis parallel to y axis	$x_0 < 0$	
	$x_0 > 0$	
	$x_0 = 0$	
Upward facing concavity	$a > 0$	
Downward facing concavity	$a < 0$	
Vertex at the origin of the coordinate system	$x_0 = y_0 = 0$ $c = 0$	 $a > 0$
		 $a < 0$
Larger opening	$0 < \|a\| < 1$ $\|b\| > 1$ $a > 0$	
Smaller opening	$\|a\| > 1$ $0 < \|b\| < 1$ $a < 0$	

Continued from Table 4

Meaningful linguistics unit	Meaningful symbolic units	Visual variables
Vertical displacement (considering upward facing concavity)	$y_V < 0$ (downward) $[a > 0]$	
$y = at^2 + ct + y_0$ $y = a\left(t^2 + \frac{c}{a}t\right) + y_0$ $y - \underbrace{\left[y_0 - \left(\frac{c}{2a}\right)^2\right]}_{y_V} = a\left(t + \frac{c}{2a}\right)^2$	$y_V > 0$ (upward) $[a > 0]$	
	$y_V = 0$ (has no vertical displacement) $[a > 0]$	
Horizontal displacement (considering upward facing concavity)	$x_V < 0$ (leftward) $[a > 0]$	
	$x_V > 0$ (rightward) $[a > 0]$	
	$x_V = 0$ (has no horizontal displacement) $[a > 0]$	

The relationships shown in Table 4 allow the verification of how changes in visual variables influence symbolic units and vice versa, an activity that is essential for the integral understanding of curves, according to the global interpretation approach of figurative properties. Another idea approached by Martins (2017), proposed and discussed by Luiz (2010) is the use of computer procedures, in this case, the Geogebra software, in order to directly obtain the parameterized curve sketching to make an analysis of the set of basic graphic units, linguistic or symbolic units with the view to lead to conversions between algebraic and graphic registers. The use of the computer procedure, according to these authors, allows some advantages related to the fast visualization of the curve and the change of scales and parameters.

Polynomial functions from the notion of infinitesimal

Aiming to problematize the curves sketching of polynomial functions of second and third degree, in high school, Pasa & Moretti (2016) and Pasa (2017) presents an alternative route for this activity and analyze its possibilities and limitations related to education and to learning under the light of the global approach to figurative properties. In Pasa (2017), the curves of polynomial functions of the second ($y = ax^2 + bx + c$) and third ($y = ax^3 + bx^2 + cx + d$) degree are sketched based on the rates of variation of the functions, understood through the notion of infinitesimal, without the formalization and rigor required in the work with the concept of limits. In this view, it is raised reflections based on the symbolic recognition basic units, on variability and graphical basic units, and more than that, on conversions between them, without the need for obtaining the algebraic function.

The rates of variation, although widely used in high school, are only worked in-depth in college, more specifically in Differential and Integral Calculus subjects, and with rigor and formalization inappropriate for high school work. Thus, it was used the *notion of infinitesimal* in the calculation of the variation rates, starting with the understanding of the average variation rate, in order to provide the global interpretation, from significant visual units, and to enable the student of this level of education to understand the required variability for curves sketching and for understanding phenomena and analysis of situations.

Thereby, the average rate of change - *ARC* of a polynomial function in an interval $[x, x + \Delta x]$, $ARC = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$; the instantaneous rate of change, identified by *IRC* (x) or more precisely[5], instantaneous rate of chance of first-order, identified by IRC_1 (x), in turn, is obtained from *ARC* (x), being Δx an infinitesimal.

In the case of the 2nd degree polynomial functions, the generic IRC_1 (x) is a 1st degree polynomial function, and an analysis (study of the sign) of this function goes beyond the comprehension of Table 1. Thus, in relation to the visual variable *inclination direction* and based on Table 1, Table 5 presents visual variable data from the tangent line to the polynomial function curve.

[5] Pasa (2017) used *RCI* (x) or RCI_1 (x) as the instantaneous variation rate of first order of a function. The "1" index is necessary when adding the idea of variation of the instantaneous rate of change, related to the concavity of a curve and represented by RCI_2 (x), or instantaneous rate of change of second-order of the function. In the case of this work, we use RCI_1 (x).

Table 5

Relationship between the visual variables of the tangent straight-line to a curve and its symbolic units.[6]

Visual Variables	Values	Corresponding symbolic units
	Ascending	$IRC_1(x) > 0$
Tilt direction of the tangent straight-line	Descending	$IRC_1(x) < 0$
	Constant	$IRC_1(x) = 0$

As the study in this perspective needs the knowledge of Calculus, it is essential to use basic units referring to relative maximums and minimums presented by Moretti, Ferraz & Ferreira (2008). In Tables 6 and 7 below, Pasa & Moretti (2016) and Pasa (2017) present these basic units – modified from Moretti, Ferraz & Ferreira (2008) – required for high school work.

Table 6

Visual and symbolic variables of a relative minimum.[7]

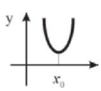
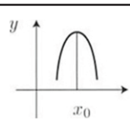
Graphic Basic Unit	Basic language unit	Symbolic basic unit
	Relative minimum at x_0. y instantaneous rate of chance of first-order changes from negative to positive signal on x_0 nearness.	$\begin{cases} IRC_1(x_0) = 0 \\ IRC_1(x_0) < 0, x \in C^-(x_0) \\ IRC_1(x_0) > 0, x \in C^+(x_0) \end{cases}$

Table 7

Visual and symbolic variables of a relative maximum.[8]

Graphic Basic Unit	Basic language unit	Symbolic basic unit
	Relative maximum at x_0. y instantaneous rate of chance of first-order changes from positive to negative signal on x_0 nearness.	$\begin{cases} IRC_1(x_0) = 0 \\ IRC_1(x_0) > 0, x \in C^-(x_0) \\ IRC_1(x_0) < 0, x \in C^+(x_0) \end{cases}$

In Tables 6 and 7 it is evident that the *relative minimum* and *relative maximum* visual variables can be analyzed and understood from the study of the IRC_1 (x) function signal. In the case of the existence of a relative minimum, the IRC_1 (x) is negative at left of the minimum (function is decreasing) and at right of the minimum (function is increasing) and at the point where the minimum occurs IRC_1 $(x_0) = 0$. The analysis of the maximum is similar.

[6] Modified from Duval (2011a, pp. 101), Pasa & Moretti (2016, pp. 9) and Pasa (2017, pp. 140).

[7] Modified from Pasa & Moretti (2016, pp. 9), Pasa (2017, pp. 141) and Moretti, Ferraz & Ferreira (2008, pp. 106).

[8] See note 6.

Another important visual variable for curves sketching is the *curve concavity,* especially for third degree polynomial functions. The upward and downward concave are related to the tangent lines increasing or decreasing angular coefficients or, rather, the IRC_1 (x) (instantaneous rates of chance of first-order function). Thus, the function has *upward concavity* on I open interval if the IRC_1 (x) *is increasing* on this interval and has *downward concavity* on I open interval if the IRC_1 (x) *is decreasing* on this interval.

The analysis of the concavity is made by the analysis of IRC_1 (x) variation, called instantaneous rate of change of the second-order function, or IRC_2 (x). According to Pasa (2017):

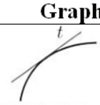
- the *curve concavity is upward* if a $IRC_2(x) > 0$, ie if a IRC_1 (x) is increasing in the range.

- the *curve concavity is downward* if a $IRC_2(x) < 0$, ie if a IRC_1 (x) is decreasing in the interval.

At the point where a $IRC_2(x)$ is null, called the *inflection point,* the concavity change occurs. The following table presents the graphic basic units and their respective symbolic units related to the concavity of a curve, related to the inclination of the tangent line, to the increasing or decreasing of the function.

Table 8

Graphic and symbolic basic units of the curve concavity of a function. (Pasa, 2017, pp. 145)[9]

Graphic Basic Unit		Linguistic Basic Unit	Symbolic Basic Unit
	(1)	t is the tangent Crescent function Negative concavity	$t{:}\ y = ax + b, a > 0$ $IRC_1 > 0$ $IRC_2 < 0$
	(2)	t is the tangent Crescent function Positive concavity	$t{:}\ y = ax + b, a > 0$ $IRC_1 > 0$ $IRC_2 > 0$
	(3)	t is the tangent Decrescent function Negative concavity	$t{:}\ y = ax + b, a < 0$ $IRC_1 < 0$ $IRC_2 < 0$
	(4)	t is the tangent Decrescent function Positive concavity	$t{:}\ y = ax + b, a < 0$ $IRC_1 < 0$ $IRC_2 > 0$

In the case of the graphical basic unit (1), of Table 8, where the line is tangent to the increasing curve, i.e. $IRC_1(x) > 0$, but a $IRC_2(x) < 0$, there is a downward facing concavity. Whereas in case (2) the line tangent to the curve is also increasing, but the $IRC_2(x) > 0$, then the concavity is turned upwards.

The visual variables presented in Tables 5, 6, 7 and 8, with the understanding and analysis of $IRC_1(x)$ and $IRC_2(x)$ make possible sketch generic polynomial functions of 2nd

[9] Modified from Moretti, Ferraz & Ferreira (2008, pp. 115).

degree, on the $y = ax^2 + bx + c, a \neq 0$ form, as presented in Table 9. So the instantaneous rate of change of the first and second-order of these functions, at any value of x, are $IRC_1(x) = 2ax + b$ and $IRC_2(x) = 2a$. Thus, $IRC_2(x)$ reveals that the parabola concavity depends on the sign of the a function parameter.

Table 9

Symbolic and graphic units of a second-degree polynomial function. (Pasa, 2017, pp. 146)

Symbolic Basic Units				Graphic Basic Units			
IRC_1	Value of a	IRC_1	Value of x	Tangent straight-line	Concavity (IRC_2)	Critical point	Curve sketching
$2ax+b$	$a > 0$	< 0	$x < -b/2a$	Decrescent	Upward (positive)	Absolute minimum when $x = -b/2a$	
		$= 0$	$x = -b/2a$	Constant			
		> 0	$x > -b/2a$	Crescent			
	$a < 0$	< 0	$x > -b/2a$	Crescent	Downward (negative)	Absolute maximum when $x = -b/2a$	
		$= 0$	$x = -b/2a$	Constant			
		> 0	$x < -b/2a$	Decrescent			

An upward-facing parabola, for example, has a minimum when $IRC_1(x) = 0$ and this occurs when $x = -b/2a$; is decreasing for values of $x < -b/2a$, since $IRC_1(x) < 0$ and increasing for values of $x > -b/2a$, where $IRC_1(x) > 0$. In addition, $IRC_2 > 0$.

In the case of third degree polynomial functions, the first and second-order instantaneous rates of change, calculated from the notion of infinitesimal, are $IRC_1(x) = 3ax^2 + 2bx + c$ and $IRC_2(x) = 6ax + 2b$. The curve sketching from $IRC_1(x)$ of the function is shown in Table 10. In this table it is possible to visualize the relations between symbolic basic units, referring to the variability of the function, more specifically to $IRC_1(x)$, and the graphic units or visual variables, referring to the tangent straight-line, to the critical points that culminate in the curve sketching. The table 11 exposes the outline of the curve from the analysis of the $IRC_2(x)$ concavity.

Table 10

Curves sketching of 3rd degree polynomial functions from the $IRC_1(x)$ analysis. (Pasa, 2017, pp. 149).

Symbolic basic units					Graphic basic units	
$IRC_1(x)$ Coef. *a*	**NR***	**$IRC_1(x)$**	**Values of**	**TL****	**Curve Sketching**	**Critical points**
$3ax^2+2bx+c$	2	<0	$\frac{-b-\sqrt{b^2-3ac}}{3a}<x<\frac{-b+\sqrt{b^2-3ac}}{3a}$	Decres		Relatives max. and min. ($IRC_1(x)=0$). Inflexion Point ($IRC_2(x)=0$)
		$=0$	$x=\frac{-b\pm\sqrt{b^2-3ac}}{3a}$	Const		
		>0	$x<\frac{-b-\sqrt{b^2-3ac}}{3a}$ e $x>\frac{-b+\sqrt{b^2-3ac}}{3a}$	Cresc		
	1	$=0$	$x=\frac{-b}{3a}$	Const		Inflexion Point ($IRC_2(x)=0$)
		>0	$x<\frac{-b}{3a}$ e $x>\frac{-b}{3a}$	Cresc		
	0	>0	$x\in R$ Sketching from $IRC_2(x)$ analysis - table 11.	Cresc		Inflexion Point ($IRC_2(x)=0$)
	2	<0	$x<\frac{-b-\sqrt{b^2-3ac}}{3a}$ e $x>\frac{-b+\sqrt{b^2-3ac}}{3a}$	Decres		Relatives max. and min. ($IRC_1(x)=0$). Inflexion point ($IRC_2(x)=0$)
		$=0$	$x=\frac{-b\pm\sqrt{b^2-3ac}}{3a}$	Const		
		>0	$\frac{-b-\sqrt{b^2-3ac}}{3a}<x<\frac{-b+\sqrt{b^2-3ac}}{3a}$	Cresc		
	1	$=0$	$x=\frac{-b}{3a}$	Const		Inflexion point ($IRC_2(x)=0$)
		<0	$x<\frac{-b}{3a}$ e $x>\frac{-b}{3a}$	Decres		
	0	<0	$x\in R$ Sketching from $IRC_2(x)$ analysis - table 11.	Decres		Inflexion point ($IRC_2(x)=0$)

* NR = Number of Roots

** TL = Tangent Line

Table 11

Concavity analysis of 3rd degree polynomial function curves. (Pasa, 2017, pp. 150)

Symbolic basic units				Graphic basic units	
IRC_2	Coef. a	Signal of IRC_2	Value of x	Concavity	Possible curves sketches
		< 0	$x < -b/3a$	Negative downward	– 
	$a > 0$	$= 0$	$x = -b/3a$	Inflexion local	
		> 0	$x > -b/3a$	Positive – upward	
		> 0	$x < -b/3a$	Positive – upward	
		$= 0$	$x = -b/3a$	Inflexion local	
	$a < 0$	< 0	$x > -b/3a$	Negative downward	–

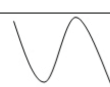
Tables 9, 10 and 11 constitute an alternative path reference to polynomial functions of the second and third degrees, pointing essential elements for the draft in advance of the global interpretation approach to figural properties using as a resource for this, the notion of variability of function.

Take the example of $y = -x^3 - 3x^2 + 9x + 6$, the same function used by Pasa (2017). The procedure of sketch this curve consists in finding an expression for the function *ARC*, for the $[x, x + \Delta x]$ range and, then determining the IRC_1 (x) and its variation, that is, the IRC_2 (x), by the notion of infinitesimals. So we have IRC_1 $(x) = -3x^2 - 6x + 9$ and IRC_2 $(x) = 6x - 6$. In this case, the values of x that nullify the IRC_1 (x) are $x = -3$ and $x = 1$.

Table 12 presents the curve sketching of the function $y = -x^3 - 3x^2 + 9x + 6$ based on the study of the signal of IRC_1 (x), indicating the critical points of the curve.

Table 12

Outline of the function curve from the analysis of . (Pasa, 2017, pp. 147)

Symbolic basic units		Graphic Basic Units		
Values of x	$IRC_1(x)$	Tangent straight-line	Curve sketching	Critical points
$x < -3$	< 0	Decrescent		
$x = -3$	$= 0$	Constant		Relative minimum in (-3,-21)
$-3 < x < 1$	> 0	Crescent		
$x = 1$	$= 0$	Constant		Relative maximum in (1,11)
$x > 1$	< 0	Decrescent		

The concavity, analyzed from the study of IRC_2 $(x) = -6x - 6$, is turned upwards when $x < -1$ and downwards when $x > -1$, being $x = -1$ the inflexion point. The critical points occur at $x = -3$, a relative minimum equal to $y = -21$ and $x = 1$, a relative maximum, equal to $y = 11$.

Trigonometric functions from the notion of infinitesimal

Another possibility briefly discussed by Pasa (2017) is the work with trigonometric functions $y = sin\ x$ and $y = cos\ x$ from the perspective of understanding their variability from the notion of infinitesimal.

7.1 Function Analysis $y = sin\ x$

The average rate of change of the function $y = sin\ x$, for the range $[x, x + \Delta x]$ is $ARC = \frac{sin(x+\Delta x) - sin\ x}{\Delta x}$ using the sine addition of two arches: $(x + \Delta x) = sin\ x.cos\ \Delta x + sin\ \Delta x.cos\ x$, one has $ARC = \frac{sin\ x.cos\ \Delta x + sin\ \Delta x.cos\ x - sin\ x}{\Delta x}$. The analysis of $sin\ \Delta x$ is carried out taking into account that Δx is an infinitesimal and using the geometric notion from a right triangle with infinitesimal Δx angle. Thus, being $sin\ \Delta x = \frac{opposite\ side}{hypotenuse}$ with the opposite side also will be an infinitesimal, which, divided by hypotenuse results in an infinitesimal thereby $sin\ \Delta x = \frac{opposite\ side}{hypotenuse} = \Delta x$.

Thus, the first order instantaneous rate of change of the sine function is $IRC_1 = \frac{sin\ x.1 + \Delta x.cos\ x - sin\ x}{\Delta x} = \frac{\Delta x.cos\ x}{\Delta x} \rightarrow IRC_1 = cos\ x$. The study of the signal and the curve sketching of the function are shown in Table 13.

Table 13

Curve sketching of the function $y = sin\ x$ *in the domain of* $[0,2\pi]$. (Pasa, 2017, pp. 152).

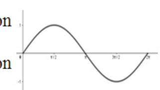
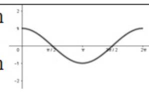
Symbolic Basic Units		Graphic Basic Units		
$IRC_1(x)$	Value of x	Tangent straight-line	Critical point	Curve sketching
> 0	$0 < x < \pi/2$ $3\pi/2 < x < 2\pi$	Crescent	Absolute minimum on $(3\pi/2, -1)$	
$= 0$	$x = \pi/2\ e$ $x = 3\pi/2$	Constant	Absolute maximum on $(\pi/2, 1)$	
< 0	$\pi/2 < x < 3\pi/2$	Decrescent		

Table 13 makes possible curve sketching the function $y = sin\ x$ for the period $[0,2\pi]$, relating the critical points and the tilt of the tangent straight-line to the study of the function signal $IRC_1 = cos\ x$.

7.2 Function Analysis $y = cos\ x$

In the case of the function $y = cos\ x$, the calculation of $IRC_1(x)$ is similar to the function $y = sin\ x$: find out the ARC for the range $[x, x + \Delta x]$, $ARC = \frac{cos(x+\Delta x) - cos\ x}{\Delta x}$ and using the cosine sum of two arcs: $cos(x + \Delta x) = cos\ x.cos\ \Delta x - sin\ x.sin\ \Delta x$, it has $ARC = \frac{cos\ x.cos\ \Delta x - sin\ x.sin\ \Delta x - cos\ x}{\Delta x}$. Being $cos\ \Delta x = 1$ and $sin\ \Delta x = \Delta x$ then the function $y = cos\ x$ has $IRC_1(x) = -sin\ x$ and the sketching is presented in Table 14.

Table 14
Sketching for the function $y = \cos x$ *in the domain of* $[0,2\pi]$. (Pasa, 2017, pp. 152).

Symbolic Basic Units		Graphic Basic Units		
$IRC_1(x)$	Value of x	Tangent straight-line	Critical point	Curve Sketching
> 0	$\pi < x < 2\pi$	Crescent	Absolute minimum on $(\pi, -1)$	
$= 0$	$x = 0$ e $x = \pi$	Constant	Absolute maximum on $(0,1)$ e $(2\pi, 1)$	
< 0	$0 < x < \pi$	Decrescent		

Tables 13 and 14 allow associations between visual variables and symbolic basic units related to the first order rate of change of sine and cosine functions. The instantaneous rate of change of the second-order ($IRC_2(x)$) was not addressed because it is not necessary when sketching said curves.

FINAL CONSIDERATIONS

Several research works on the curves sketching from the perspective of the global interpretation of figurative properties are being carried out with the objective of proposing resources that enable the articulation between graphic and algebraic record from significant elements, which, as Duval (2011) points out, are essential for learning. We presented in this article studies with possibilities of work in the high school.

In the case of **the related function,** Duval (2011) highlights the relationships between the coefficients of the algebraic expression $y = ax + b$ and the visual variables of the graph with the tilt direction, the tracing angles with the axes and the tracing position in relation to the origin of the vertical axis. Using the square complementation treatment in the algebraic writing of **the quadratic function** ($y = ax^2 + bx + c$), Moretti (2003) discusses how to maintain the relationship between visual representation variable and symbolic units, using the translation function. The **trigonometric, exponential** and **logarithmic function** curves sketching were proposed by: Silva (2008), Corrêa & Moretti (2014); using as resources the translation and symmetry along with the significant units of the algebraic expression represented by the coefficients. Menoncini & Moretti (2017) presented possibilities for working with **modular functions** using coefficients of algebraic representation as symbolic basic units. Martins (2017) studies the **parametric equations** of the **line** and the **parabola,** whether they are functions or not, relating visual variables with the coefficients of parametric equations. In the case of straight lines, the visual variables in question are: the direction of inclination, the angles with the axes and the position on the axes. The parabolas were studied relating the visual variables: concavity, vertex and axis of symmetry. In addition, Martins (2017) uses the Geogebra software that makes it possible to directly obtain the graphical representation of the curve and thus verify the figurative properties and their relationships with symbolic units.

Pasa & Moretti (2016) and Pasa (2017) suggest an alternative path for working with **second degree, third degree polynomial functions,** and **sine** and **cosine trigonometric functions** based on the study of the instantaneous rate of change of the function signal found from of the notion of infinitity. For this, it uses visual variables such as tangent straight-line inclination, relative minimum and maximum, concavity and inflection point, associated with symbolic units of the rates of change.

Understanding the theory of semiotic representation registers allows the mathematics teacher a deep understanding of the cognitive activities required for the learning of mathematical objects, which directly influences their pedagogical actions in the classroom and which will provide this learning. In the case of curves sketching, Duval's approach to global interpretation of figurative properties requires the transition between different curve registers from the identification and articulation of symbolic basic units of algebraic expression and graphic basic units, actions that are not trivial and that become more difficult due to non-matching between registers. The presented works are pedagogical possibilities that allow to overcome learning obstacles and increase the degree of semantic congruence between the records of a function or equation.

According to Pasa & Moretti (2016), the discussions and reflections on the curves sketching are pertinent, mainly due to the importance of this activity in current times, as a way of representing phenomena in all areas of knowledge and everyday situations. Moreover, due to the difficulties presented by students in the activity of sketching and interpreting curves, which are due, among other factors, to a teaching approach that prioritizes the "point to point" graphing and the passage from the algebraic register to the graph only, not valuing the understanding of the semiotic correspondence rules between the graphical representation register and the algebraic expression register.

AUTHOR CONTRIBUTION STATEMENT

B.C.P. and M.T.M. conceived the idea of the article. B.C.P. conducted the research on papers published about this article subjects, organized the collected data and, together with M.T.M., proposed working with infinitesimals in high school to curve sketching of polynomial functions of 2nd degree, 3rd degree and the sine and cosine trigonometric functions. B.C.P. and M.T.M. also discussed the results together and contributed to the final version of the article.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article because no new data was created or analyzed in this study.

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