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# The Movement of Teachers' Theoretical Thinking about the Concept of Fraction and the Meaning Attributed to Teaching Materials: An Example of Multiplication with Fractions

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## ABSTRACT

**Background**: In recent years, many states and municipalities have adopted official teaching materials to guide teacher practice. However, the relationship between the use of these materials and the continuing education of in-service teachers has been little investigated. **Objective**: To analyse how the development of the theoretical thinking of teachers about the multiplication of fractions impacts the choice, use, or adequacy of teaching materials. **Design**: The theoretical and methodological foundation is based on the historical-cultural perspective seeking, in the investigation of human activity, to understand the development of theoretical thinking. **Setting and participants**: Continuing education with a group of 11 mathematics teachers from the state public network of São Paulo. Learning triggering situations have been proposed to overcome discretisation in the approach to fractions. Data collection and analysis: Data were collected through video recordings, written material produced by teachers and registers in a field diary. Two isolated were organised for analysis "Movement of theoretical thinking about fractions" and "Didactic material as a mediator of the teacher's action." **Results**: As they theoretically understand the meaning of multiplication between fractions, teachers overcome the mechanisation of rules and begin to reveal an understanding of the interrelationship of arithmetic, algebraic, and geometric elements in teaching. **Conclusion**: The development of aspects of theoretical thinking about the multiplication of fractions allowed teachers to organise teaching in order to make explicit the essential and theoretical relationships of the concept. Didactic material is understood as mediation in the teaching process and its use is no longer seen as an end in itself.

**Keywords**: Theoretical thinking; Sense; Teaching materials; Teacher education; Multiplying Fractions.

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**O Movimento do Pensamento Teórico de Professores sobre o Conceito de Fração e o Sentido Atribuído aos Materiais Didáticos: Um Exemplo de Multiplicação com Frações**


## RESUMO

**Contexto:** Nos últimos anos, muitos estados e municípios têm adotado materiais didáticos oficiais para orientar a prática do professor. No entanto, a relação entre o uso desses materiais e a formação continuada dos docentes em serviço tem sido pouco investigada. **Objetivo:** Analisar como o desenvolvimento do pensamento teórico de professores sobre multiplicação de frações impacta na escolha, utilização ou adequação dos materiais didáticos. **Design:** O fundamento teórico metodológico baseia-se na perspectiva Histórico-Cultural buscando, na investigação da atividade humana, compreender o desenvolvimento do pensamento teórico. **Ambiente e participantes:** Formação continuada para um grupo de 11 professores de matemática da rede pública estadual de São Paulo. Foram propostas situações desencadeadoras de aprendizagem visando superar a discretização na abordagem das frações. **Coleta e análise de dados:** Os dados foram coletados por meio gravações em vídeo, material escrito produzido pelos professores e registros em diário de campo. Foram organizados dois isolados de análise "Movimento do pensamento teórico sobre frações" e "Material Didático como mediador da ação do professor". **Resultados:** À medida que compreendem teoricamente o sentido da multiplicação entre frações os professores superam a mecanização das regras e passam a revelar compreensão da inter-relação dos elementos aritméticos, algébricos e geométricos no ensino. **Conclusão:** O desenvolvimento de aspectos do pensamento teórico sobre multiplicação de frações permitiu aos professores organizarem o ensino de modo a explicitar as relações essenciais e teóricas do conceito. O material didático é entendido como mediação no processo de ensino e seu uso deixa de ser visto como fim em sim mesmo.



**Palavras-chave:** Pensamento teórico; Sentido; Materiais Didáticos; Formação de Professores; Multiplicação de Frações.


## INTRODUCTION

The social function of the school and its players is often the object of discussion in the face of social transformations and their relationship with changes in qualification required by the labour market. Almost naively, we often seek to establish a linear relationship between the labour market demands and teacher education, which, in turn, has gained prominence as an object of research in Brazil and the world (Freitas, 2002; Libâneo, 2004; Nunes, 2000;

among others). In this context, it is necessary to reflect and analyse critically how this teacher education is taking place.

Taking as a context the proposals for teacher education of the state of São Paulo and, in particular, the materials that teachers use in their daily practice (São Paulo, 2014a, 2014b, 2014c), this article discusses the use of the so-called *Cadernos do Professor* (teacher's notebooks) in mathematics, as a training strategy. For this, we present an excerpt from the master's research that investigated the relationship between the use of teaching materials, in particular, the materials that make up the official curriculum of the state of São Paulo, and the development of theoretical thinking of mathematics teachers. For this article, we will highlight the movement of teachers' theoretical thinking on the multiplication of fractions, in dialectical relation to the meaning that teachers attribute to the official materials[1] of the state of São Paulo.

To this end, we begin the article by presenting the context of the constitution of the research object. The following are the theoretical references about the concepts of activity (Leontiev, 1978) and theoretical thinking (Davidov, 1988). We will describe the methodological path for the development of the formative experiment and bring an example of a learning triggering situation, from the concept of teaching guiding activity (Moura, 1996), followed by strategies for data collection and analysis. Finally, we will present excerpts from the analysis of the research data (Romeiro, 2017) and our conclusions.

## TEACHER EDUCATION AND CONSTITUTION OF THE RESEARCH OBJECT

School has long been invaded by demands for qualification arising from the labour market, which requires workers with skills that cannot be developed at work in the short term. Thus, it seeks to assign the educational system the responsibility for qualifying individuals for work. By doing this, the capital exempts itself from the time and investment needed for the development of such skills (Nunes, 2000). Although school should dialogue with social

[1] The official materials are the handouts that make up the Official Curriculum of the State of São Paulo. They contain guidelines for teachers (*Caderno do Professor*) and tasks for students (*Caderno dos Alunos*).

reality and the world of work, the relationship between school and the world of work should be looked at thoroughly, so that there is no inversion between ends and means of the educational process (Moretti, 2007). That is, by understanding the objective of formal education as being to prepare the individual for "the world of work, which is mediation and, therefore, particular, is taken as universal. Labour, as a human activity, a social mediation needed for the objectification of the individual-human gender relationship, is replaced by work, synonymous with employment, and assumed as an end" (Moretti, 2007, p. 46). We understand that the inversion occurs as the qualification for the world of work is taken as the main objective of education, and not as a consequence of a process that aims at a generic and critical human formation.

We sought to counter the implications of this relationship of subsumption between work and education for teacher formation. This research assumes work as a constitutive category of the human being and the concept of activity (Leontiev, 1978) as the organising principle of teaching practice. Considering that the research was conducted with teachers from the public school system of the state of São Paulo, we present below some of the principles of the documents regulating teacher education in this context.

Following the logic of developing skills to meet the demands of the labour market, the Ministry of Education (Brasil, 2002) states that it is essential to link teacher education with the qualification expected of students to meet the demands from society and the world of work. This document also adds evidence that teacher education is failing to ensure the preparation expected from students, being necessary to invest in continuing education to correct the gaps in initial education.

In the state of São Paulo, as of 2008, the continuing education of teachers in service envisions the implementation and consolidation of the *Proposta Curricular do Estado de São Paulo* (Curriculum Proposal of the state of São Paulo). This proposal presents a single and mandatory curricular reference for all schools in the state of São Paulo that includes the development of competencies and skills. Its purpose was to improve the results of external assessments.[2] It was also considered a form of democratisation of the teaching

[2]The focus was on the result of the *Sistema de Avaliação da Educação Básica* - SAEB (Basic Education Assement System), which began in the late 1980s, and the *Sistema de Avaliação e Rendimento Escolar do Estado de São Paulo*- SARESP (School Assessment and Performance System of the State of São Paulo), created in

and universalisation of access, as the classroom contents and activities would be the same all over the state. In 2010, this proposal for a unified curriculum became the *Currículo Oficial Paulista* (Official Curriculum of São Paulo), implemented in the final years of elementary school and high school all through the state of São Paulo (São Paulo, 2012).

For its consolidation, the official organs prepared a set of documents to guide the network on the methodologies and strategies to be adopted in the state public schools. They are *Caderno do Gestor* (school manager's notebook), *Caderno do Professor* (teacher's notebook), and *Caderno do Aluno* (student's notebook). *Caderno do Gestor* provides methodological guidelines for actions with continuing teacher education, management, and implementation of the official curriculum. *Caderno do Professor* offers methodological guidance on classroom pedagogical practice, the tasks to be performed by the students, and the assessment processes to be adopted. *Caderno do Aluno*, in turn, brings the concise conceptual definition of the contents, and tasks to be performed based on given models. The Secretariat of Education of the state of São Paulo considered these guidelines essential in classroom practice, so teachers and students were required to use them.

Considering the official guidelines for teacher education linked to the development of competencies in line with the demands of the labour market and considering the possible impacts of these guidelines on the social function of the school and teaching work, many researchers have mobilised to study the initial and continuing education of teachers assuming a theoretical perspective that understands education as a process of humanisation and the teacher as a fundamental agent in the development of the maximum psychic and human potential of students, through the organisation of teaching and choice of mediating instruments (Cedro, 2008; Moretti, 2007; Moura, 1996; among others).

Inserted in this movement and considering the context of the proposal of teacher education underlying the unified official materials that make up the official curriculum of São Paulo, we question how much the use of such materials favoured the work of the mathematics teacher and, consequently, the student's learning for this human formation. Starting from a theoretical

1996, which until then had results far below the targets set by international bodies, such as the OECD.

understanding of human formation based on the activity theory (Leontiev, 1978) and the developmental teaching theory (Davidov, 1988), we developed a research that sought to answer whether teachers would change their use of such materials as their knowledge of the concepts to be taught changed and, particularly, whether the development of theoretical thinking (Davidov, 1988) about certain content impacts on teaching organisation and the use of official materials. We chose to address the concept of fraction in this research because of the difficulties teachers and students reveal when dealing with it in the final years of elementary school (Rosa, Holbold, Bernardo, Corrêa, & Inácio, 2013). Below, we present elements of the theoretical foundation that guided the development of the research.

## HUMAN ACTIVITY AND THE DEVELOPMENT OF THEORETICAL THINKING

The historical and cultural theoretical framework argues that the subject becomes human through human work in social and cultural relations, producing human needs and instruments that can meet them (Vygotsky, 2010). From the creation of instruments to meet the essential needs or those generated by the subject, man controls nature and modifies it at the same time as it is modified, differentiating himself from other animals. It is in this dialectical relationship that social and cultural productions develop and transform throughout the history of human beings.

In this theoretical context, the concept of competence, addressed in teacher education, is resignified in relation to common sense and official guidelines. Rather than emphasising the development of competence that is associated "with the individual characteristics of the subjects, with adaptation to social reality and focused on practical activity" (Moretti, 2007, p. 77), we come to understand that competence is built "in a social process that does not dispense with specific fundamental knowledge to be appropriated by the subjects through a school education intentionally organised for the development of theoretical thinking in conjunction with social practice, and not subsumed to it" (Moretti, 2007, p. 186).

In this sense, we wanted to structure a continuing education course that would meet the needs of the participating teachers to qualify and not only to meet the impositions of the central bodies. Like Leontiev (1978), we understand

that every human activity is born from a biological, intellectual or spiritual need aimed at an object, in a motivated and mediated way. The coincidence between motive and the object that materialises the necessity characterises the activity. Thus, by structuring the training proposal, we sought to create conditions for teachers to engage in the activity so that they could understand the meaning of their work and the "continuing education as a space for permanent development of knowledge, reflection, and strengthening of their work, which provides the formation of the integral and omnilateral man in a dialectical relationship between work and necessity, that is, the teaching work articulated with the teacher's activity" (Romeiro, 2017, p. 28).

Thus, akin to the concept of activity, the concept of theoretical thinking also becomes relevant, since we intend to establish relationships between the development of teachers' theoretical thinking and the use of teaching materials. Davidov (1988) attributes distinct paths to the development of empirical and theoretical thinking. For the author, empirical thinking acts from the perspective of formal logic, in the classification of objects by their apparent and particular characteristics. Theoretical thinking, on the other hand, operates less with practical-sensory representations, and operates more with the concept because it has a generalising character that includes the internal and external relations of the object of knowledge. In this context, thinking theoretically about a concept is "to understand it in its essence within this historical process, verifying the contradictions and non-similarities in a concrete-abstract-concrete movement" (Romeiro & Moretti, 2016, p. 3). Essence is understood as the internal connection that has the genetic basis and determines all other particular specificities of the whole, overcoming appearance. The concrete-abstract-concrete movement in the development of theoretical thinking starts from the apparent concrete, and through the mediation of the abstract, returns to the real concrete considering all relations beyond appearance, both in general and in particular situations.

Davidov (1988) states that it is important that school subjects foster the development of theoretical thinking and, in the same direction, we understand that it is necessary that, through disciplines, the school develops or brings the students closer to theoretical thinking to solve practical or non-practical problems intentionally and consciously, not mechanically, from repetitions of models. In our research with teachers, our challenge was to create conditions to follow the development of theoretical thinking and the implications of this

process for the use of teaching materials. In the following item, we present the methodological organisation aimed at meeting this challenge.

## THE METHODOLOGICAL MOVEMENT OF THE RESEARCH

To follow the development of teachers' theoretical thinking, we relied on the method of dialectical historical materialism, which, according to Martins (2006), seeks to unveil and analyse the totality of the object in contradiction, in movement. As a strategy to monitor the phenomenon, we used the concept of formative experiment (Davidov, 1988), which creates possibilities to investigate the psychological changes of the individual to overcome the appearance in the education process. The characteristic of this method in research is the direct intervention of the researcher in the actions developed during the experimental investigation. For Cedro and Moura (2010), during the experimental investigation it is important to privilege the interaction and communication between the research subjects for the planning of future actions developed by the researcher.

As a theoretical and methodological resource, we resort to the concept of teaching guiding activity - TGA (Moura, 1996), which allows the teachers to structure their action intentionally, seeking learning triggering situations - LTS, that enable students to face the need for the concept, considering the dialectical relationship between the needs for teaching and learning and highlighting the essence of the concept to be taught and learned. This theoretical and methodological resource has as its principle the collectivity of the subjects in activity and presents itself as a unit between teaching and learning.

The development of the LTS is a complex process and requires the study of historical aspects of the concept, since the "triggering problem or problem-situation, in the teaching guiding activity, has as its essence the need that led humanity to the construction of the concept" (Moretti, 2007, p. 97). Thus, the development of the LTS for our research demanded the study of historical aspects of the concept of fraction.

The first registers about the concept of fraction indicate the human need to measure things (Boyer, 2010). Caraça (1989) says that the need for the measure arose in the relationship of private property, economy, and obligations

to the State, such as tax payment in Egypt. Measuring with strings often did not result in an integer quantity which implied a dilemma since, until then, there was no need for numerical representation of parts of an integer. According to Caraça (1989, p. 37), this dilemma was overcome through a mental operation called the denial of the denial, which allowed "building the new number – fractional number - that came to build the new part of the generalised field," i.e., through the denial of the impossibility of dividing integers, the need for a new numerical field in which it was possible to fractionate the whole part arose. Still, according to Caraça (1989), it is possible to infer that the need for the concept of fraction has its origin in the comparison of continuous quantities and their relations, so that the unit of measurement becomes a mediator of the process of comparison between quantities of the same nature, involving the link between multiplicity (how many times the unit is repeated in the whole) and divisibility (number of times the unit fits in the whole without leaving remainders).

In this sense, learning triggering situations[3] were organised in such a way that teachers could overcome the traditional form of fraction teaching that favours discrete quantification and the restricted meaning of fraction as a part-whole. This traditional view of fraction teaching reduces the approach of this concept to counting situations, in which how many parts the whole has been divided into and how many of those parts were taken are counted, representing this relationship through a graph (pizza, chocolate, etc.) or arithmetic means (numerator and denominator). This form does not allow the expansion of knowledge to another numerical field, i.e., in this form of discrete approach, the field of natural numbers is sufficient to solve a problem, enabling the development of empirical and apparent knowledge about fractions.

The learning triggering situations proposed to teachers aimed to overcome discretisation in the approach to fractions, and enable them to meet aspects of the essence of this concept, such as continuous quantification through measurement (Boyer, 2010; Caraça, 1989), some different meanings of a fraction (part-whole, fraction as a number, and fraction as a quotient), and equivalence and multiplication operations. In this article, we present the data referring to the movement of development or approximation of elements of theoretical thinking involving the multiplication operation, seeking to establish

[3] The entirety of learning triggering situations are presented in Romeiro's dissertation (2017).

relationships between this development and the changes in the meaning attributed by teachers to official materials.

The formative experiment took place through an extension course for mathematics teachers from the state public schools of the city of Guarulhos, authorised by the Secretariat of Education in partnership with the Federal University of São Paulo.[4] Its objective was to enable teachers to appropriate the logical-historical elements of the production of the concept of fraction, through learning-triggering situations. The theoretical and methodological resources of learning triggering situations used in the extension course were problem-situations (Moretti, 2014) and virtual history (Moura, 1996). Virtual history involves characters in a situation that brings aspects of the essence of the historical necessity of the concept. This history is not necessarily factual history, but it has as its principle to create the need to produce the concept and involve the subject who learns in the production of the solution to the problem (Moura, 1996).

The course consisted of ten face-to-face meetings on Saturdays, of four hours each, and one meeting with a proposal for distance production. Eleven mathematics teachers who worked in the final years of elementary school of the state of São Paulo participated, specifically in the *Diretoria de Ensino Guarulhos Norte* (Guarulhos North Board of Education) in the city of Guarulhos, where the face-to-face meetings took place. The teacher educator in this process had the roles of researcher and educator.

In addition to the learning triggering situations involving aspects of the logical-historical movement of the concept of fraction, such as overcoming the discrete count for continuity, including the arithmetic interrelationship (representation of the number), algebraic (properties) and geometric (numerical line), teachers were also proposed to produce learning triggering situations for their students at different moments of the formative experiment. For this, different pedagogical resources (books, internet, and official materials) were made available for consultation. Teachers could also seek other materials they considered appropriate. This movement to develop the LTS supported the analysis of the relationship between the development of the teacher's

[4] The research was duly submitted to the Research Ethics Committee of (UNIFESP), approved by Opinion No. 1,373,032 and all teachers signed the Informed Consent Form.

theoretical thinking on the concept of fraction and the choice of mediating instruments, including the use or adequacy of official materials for teaching organisation.

The data collection instruments involved semi-structured interviews (SI), direct observation of the researcher (DO), individual conversations between the researcher and the teachers surveyed (IC), formative meeting (FM), and the registration of proposals for solving learning triggering situations (RP). The material resources used in data collection included camcorders, recorders, and field diaries. The data presentation in this article are in following order: teacher's name of the teacher, acronym of the instrument used for data collection, and the number related to the formative meeting. All teachers signed an informed consent form, and the names presented are fictitious, ensuring anonymity in the research.

The data analysis was organised from two isolated: "*Movimento do pensamento teórico sobre frações"* (Movement of theoretical thinking about fractions) and "*Material Didático como mediador da ação do professor"* (Teaching Material as a mediator of the teacher's action). According to Caraça (1989), the isolated aims to understand the totality of reality, from excerpts of situations that reflect it. To reveal the isolated, we use the concept of episodes that, according to Moura (2000), is what displays the nature and quality of the isolated. The two isolated materialise a unit that sought to show how theoretical thinking about fractions impacts on the meaning attributed by teachers to the use, choice, or adequacy of teaching materials in organising teaching.

The first isolated, *Movimento do pensamento teórico sobre frações* (Movement of theoretical thinking about fractions), had four episodes and aimed to verify the movement of overcoming empirical thinking through the development of theoretical thinking about fractions. The second isolated, "*Material didático como mediador da ação do professor*" (Teaching Material as a mediator of the teacher's action) consists of two episodes and sought to understand the relationship between the development of teachers' theoretical thinking on fractions and the meaning attributed to the official materials that made up the Official Curriculum of São Paulo.

## RESULTS AND DISCUSSION: THEORETICAL THINKING ABOUT MULTIPLICATION OF

## FRACTIONS AND THE MEANING ATTRIBUTED TO OFFICIAL MATERIALS

To achieve the objective of this research, our analysis movement was to investigate, in the data collected during the training experiment, the impacts of the development of theoretical thinking on the concept of fraction in the choice, use and adequacy of mediating instruments, especially the official materials that make up the official curriculum of São Paulo.

During the meetings, as they were solving the learning triggering situations and approaching the theoretical thought about the fractions, the teachers made relations with the materials used in the classroom, including the official materials, but demonstrated a different way of thinking, manifesting, in their speeches and productions, a more conscious and critical understanding of the content and use of this material.

For this article, we chose to bring the analysis and discussion of the movement of approximation of teachers to theoretical thinking about the multiplication of fractions, and how this approximation impacted the meaning attributed by teachers to official materials, which were already present in their pedagogical routine.

Before addressing with teachers the LTS on multiplication of fractions, we asked how they used to start working with students on this operation. Usually, many teachers understand and present to students the operation of multiplication of fractions through the rule "numerator times numerator, denominator times denominator," because they find the rule is a more practical way to present the operation to students, and because they found it difficult to transit through other approaches, such as geometrics. In this direction, teachers Vitor and Sara presented their way of approaching the operation with the students as follows:

> *Vitor:* I start as simple as possible. I give the two fractions and stress: it is the simplest of the four operations. It is numerator with numerator and denominator with denominator and you multiply as usual. That's the way I work. And when there's an integer, I put one underneath for them to see that it's also a fraction. This is so as not to make the mistake of multiplying the numerator and denominator by the same number.

> *Sara:* I start with integers: 2 +2+2, this equal to what? 3x2. Now let us have the fraction: $\frac{1}{2} + \frac{1}{2} + \frac{1}{2}$ it is the same thing as $3.\frac{1}{2}$. [...] I get the concept from them. [...] When I say to them: How do we do it, when it is $\frac{1}{2} . \frac{1}{3}$? And they answer that it is the same thing as the other with equal portions, you multiply the top one by the top one and the bottom one by the bottom one, numerator with numerator and denominator with denominator [...].
>
> (FM, 8).

Although teacher Sara tries to relate the multiplication operation with the addition of equal portions, she cannot explain the generalised form of this operation, especially when it comes to multiplication between two fractions, returning the rule as presented by teacher Vitor. At this initial moment, teachers did not bring any specific material for the presentation of multiplication of fractions, only the exposure of the arithmetic model with the logical rule of this operation.

We understand that this form of exposure demonstrates the appreciation of aspect based on formal logic through the model and mechanised rule, without understanding the importance of highlighting internal and non-apparent aspects related to geometric, arithmetic, and algebraic representations, which, according to Rosa (2012), are essential for the development of theoretical thinking. Moreover, this form of presentation does not put students in motion or learning activity but instead puts them in a peaceful situation of model reproducer in particular situations.

For Davidov (1983), this form of presentation based on formal logic brings many subsequent difficulties to students, since this approach involves only the classification of particular or singular problems based on models and rules, and, in many cases, can make students claim to be impossible to solve some problem-situations involving the concept. Teacher André showed he perceived this difficulty in the way he approaches operations with fractions:

> Students confuse operations, right? They say, is it to do it in line? Is it to do in X?

> I think if they (students) understand the geometric representation, they will know how it works, right? Although it is a more difficult procedure, he will understand the concept more easily (André, FM, 8).

According to Caraça (1989), the multiplication of fractions maintains all the properties of integers, i.e., the sum of equal portions. However, when we talk about multiplication between two fractions, it is necessary to analyse a little more the relationships and transformations that this operation causes in its result, the conclusion and generalisation not being so straight forward. Davidov (1988) says it is important to introduce students to the abstract essence of mathematics and harness its theoretical strength in all its internal and external relations. In other words, the student must understand the logical-historical path of production of abstract forms of mathematics, following the concrete-abstract-concrete movement, in which the point of arrival is a concrete that goes beyond the apparent and particular understanding of the concept (mechanised rules), but that all the internal and external relations of the movement are present. This is its theoretical strength. As a way of harnessing this theoretical force described by Davidov, teachers were presented with a virtual story called "Egyptian Agriculture," which deals with the multiplication of fractions in all their relations. This learning triggering situation consists of the following:

**Figure 1**

*Learning triggering situation "Egyptian agriculture"* (Romeiro, 2017, pp. 140-141)

**Egyptian agriculture**

Agriculture was the main source of the ancient Egyptian economy. The most important crops of the time were barley and wheat. The lands on the banks of the Nile River were the most coveted among farmers, because once a year the Nile flooded the soil, making it much more fertile for planting and harvesting.

Azibo's land was not near the banks of the Nile, but used barley and wheat farming as a source of income and livelihood for his family. Azibo planted barley and wheat in the part where he did not plant the garden (planted part).

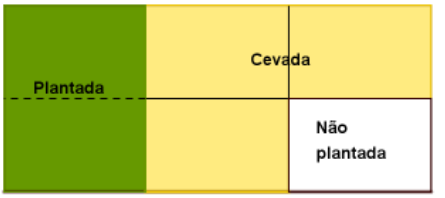
Knowing that he planted the barley in ¾ of the unplanted part, in what part of the land (the whole lot) is the barley planted?

Graph the problem resolution, write the arithmetic expression of the problem, and represent the operation and the result on the numerical line.


The graphic solution was quickly found by teachers. But relating this graphic representation to arithmetic and geometric representation was a hindrance. The teachers initially found the product immediately $\frac{3}{6}$, as shown in figure 2, and questioned how they could find the product $\frac{6}{12}$ to generalise the multiplication rule.

**Figure 2**

*Solution found by the teachers of the LTS "Egyptian agriculture"* (Romeiro, 2017, p. 142)




*Sara:* How does the student deduce that $\frac{3}{4}\frac{2}{3}$ is the same thing as $\frac{3.2}{4.3}$? He did not realise it visually, he obtained $\frac{3}{6}$. You say, multiply this with this (numerator with numerator) and this with this (denominator with denominator), and you obtain $\frac{6}{12}$. Then, you say that now it is just simplifying it, and you obtain $\frac{3}{6}$. But then you're giving it resolved. He (the student) is not getting the answer.

*Isabela:* What if you say about half of it?

*Sara:* He's still not deducting it. You're giving a situation kind of resolved.

*Isabela:* Doesn't the student know yet that it is to multiply the numerator and the denominator?

*Sara:* He doesn't know it's to multiply. So, only if you give it ready. Well, I split $\frac{3}{6}$ in half, which is the same thing as $\frac{1}{2}$, but it's also the same thing as $\frac{6}{12}$. Only if you give the approximate path.

*Isabela:* When we're going to teach multiplication of fractions, they've seen addition and subtraction. So they already know they need to find the equivalents. Only if we follow the same path, too.

*Sara:* I don't think you have to do that. I'd follow this path of giving kind of ready. Then it would represent it on the numerical line.

(Dialogue between teachers, DO, 8).

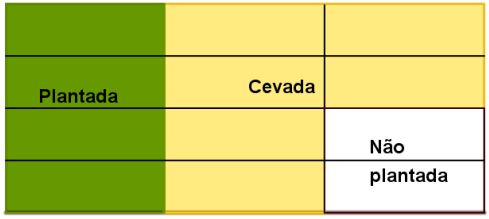
As the excerpt shows, the teachers' speech already reveals a concern for searching the logical-historical relationship of the concept of fraction and its operations, giving evidence of overcoming the appreciation only of the mechanised rule and recognising the limits of solving particular problems involving this concept.

However, to generalise the rule of the multiplication of fractions, the teachers had to leave the field of external appearances to analyse the internal relations of the problem, reaching abstraction and theoretical generalisation, that is, reaching the general form of resolution of multiplication of fractions. However, the teachers could not achieve this abstraction, because they were based on the crystallised forms of graphic representation, typical of traditional teaching, which values generalisation through graphic or arithmetic examples and pre-established rules. This excerpt corroborates the understanding that empirical thinking is a hindrance to the development of theoretical thinking of the concept (Davidov, 1988).

Seeking to overcome this empirical understanding, we chose to share dialogically with teachers another possibility of representing the problem. Initially, the resolution shared with them followed the same path of representation of the fraction they had proposed, recognising the parts corresponding to the planted area of the land and the non-planted area: $\frac{1}{3}$ and $\frac{2}{3}$, respectively. However, the form of representation changed in relation to what the group had done, when the teacher educator/researcher proposed to understand the multiplication of fractions as being part "of" a part, i.e., $\frac{3}{4}$ of $\frac{2}{3}$, not in a particular way of a part of the land, but involving the complete land, as shown in Figure 3.

**Figure 3**

*Graphic representation of the multiplication operation shared with the teachers* (Romeiro, 2017, p. 143)




From this way of thinking the multiplication of fractions, as a part of a part, Professor Sara said:

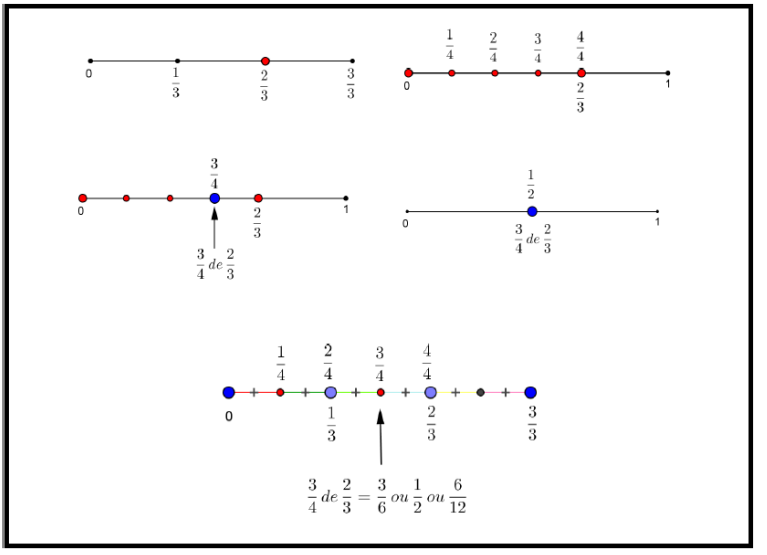
Thinking in this way, yes, it came right out $\frac{6}{12}$. [...] now you can deduce that $\frac{3}{4}$ of $\frac{2}{3}$ is $\frac{6}{12}$. Then I can get to that multiplication that $\frac{3 .2}{4 .3} = \frac{6}{12}$. Then I work the equivalents, including the irreducible $\frac{1}{2}$ (Sara, FM, 8).

The graphical representation found by teachers and the one presented by the teacher educator/researcher are correct, however, it is simpler to understand and reach the abstraction and theoretical generalisation of the rule of operation with fractions from the understanding of internal and external relations, that is, that the multiplication of fractions means "part of a part", thus establishing the algebraic relationship: $\frac{a}{b} . \frac{c}{d} = \frac{a . c}{b . d}$. This form of generalisation involves the arithmetic (numerical representation and operations), algebraic (property) and geometric (graphical representation) interrelationship, in general or particular situations.

To relate the result of multiplication between the fractions and their simplified arithmetic representation, i.e., that $\frac{3}{4} . \frac{2}{3} = \frac{6}{12}$ or $\frac{1}{2}$, teachers were asked to perform the multiplication on the numerical line. After understanding the meaning of multiplication as "part of a part," teachers multiplied faster and with improved understanding, as shown in Figure 4 below:

**Figure 4**

*A solution of the LTS "Egyptian agriculture" carried out by the teachers in the line of the real numbers* (Romeiro, 2017, p. 145)




In this situation presented, the rule of the multiplication of fractions becomes a consequence of the process of knowledge development, a generalisation of this movement, and not its starting point, as it appears in education based on formal logic.

The multiplication of fractions is addressed in the official material of the state of São Paulo, in volume 1 of the *Caderno do Professor* of the 6th grade (São Paulo, 2014a, pp. 50-51), involving multiplication of an integer by a fraction, and volume 1 of the *Caderno do Professor* of the 7th grade (São Paulo, 2014b, pp. 27-32), involving multiplication between different fractions. In this material, the teacher is oriented to approach the operation involving geometric representation from the expression "part of a part," converging with the logical-historical form of production of the operation. An example of a task contained in *Caderno do Professor* to address the concept of multiplication of fractions is as follows:

**Figure 5**

*Task corresponding to the multiplication operation approach with fraction in Caderno do Professor* (São Paulo, 2014b, p. 29)

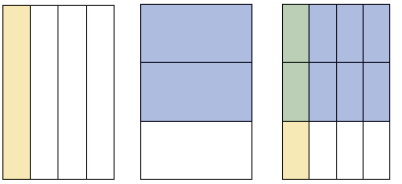
Um pintor utiliza $\frac{1}{3}$ de uma lata de tinta e, em seguida, $\frac{1}{4}$ do que restou de tinta na lata.

**a)** Escreva uma operação com frações que represente corretamente a fração da lata que foi utilizada da última vez.

Ao utilizar $\frac{1}{3}$ da lata, restam $\frac{2}{3}$.

Como da última vez utilizou-se $\frac{1}{4}$ de $\frac{2}{3}$, a operação procurada é $\frac{1}{4} \cdot \frac{2}{3}$.

**b)** Realize a operação indicada no item anterior utilizando "barrinhas".



$\frac{1}{4} \cdot \frac{2}{3} = \frac{2}{12}$

It is possible to identify that the approach of the official material on the multiplication of fractions converges with the triggering situation of learning "Egyptian agriculture" proposed to teachers. Professor Sara seems to have identified this relationship:

In the handout, she works like this, [...] the problem of the paint can. John Doe used half of the can on the first day. The other day, he used $\frac{2}{3}$ of what was left, i.e., $\frac{2}{3}$ of the half. If he used half in one day, there's half left for the next. If we work with the handout, we must do the geometric representation, just what we are doing now (Sara, FM, 8).

From the approximation of the elements of theoretical knowledge on the multiplication of fractions, the teachers made relations with the official material, including the form of teaching organisation and the use of didactic material.

*Caderno do Aluno* does not start straight away with multiplication to represent. It starts with half a third. So, it starts with the representation of a third and takes half of this third and deduces that it gives a sixth. Then, after that, it becomes multiplication. [...] This makes sense when I speak half a third (Sara, FM, 8).

Given the data, here expressed in the speech of teacher Sara, we can see that the official material (teacher's notebook) starts to be analysed more consciously and critically as teachers demonstrate a more theoretical understanding of the concept addressed. To analyse more carefully this relationship between the approximation of the theoretical elements on the multiplication of fractions and the use of didactic material as a mediator of the activity of the teacher and the organisation of teaching, we asked the teachers to produce in a group a learning triggering situation on the multiplication of fractions. In this production, teachers could seek the mediating instrument that best fits the intentionality of teaching.

Teacher Vitor's group proposed the following learning triggering situation on the concept of multiplication of fractions:

**Figure 6**

*LTS prepared by teacher Vitor's group* (Romeiro, 2017, p. 170)

*A team of engineers, looking for an area for a building, found a lot, according to the figure. On this lot, they managed to build a tower with parking for cars and motorcycles. Knowing that* $\frac{1}{4}$ *the parking lot will be reserved for motorcycles, what is the total part of the lot that corresponds to the motorcycles?*

> We sought to make an LTS [...] in which the student will have to look for all the information in the text and in the figure to solve the problem. The goal is to make the subdivisions to get to the part that is reserved for motorcycles and complete the multiplication property with them (students) (Vitor, FM, 10).

This proposal approximates the LTS "Egyptian agriculture," as well as the way of explaining the subject in the official materials of the state of São Paulo, in which constitutive elements of theoretical thinking are present, as shown in figures 3 and 4, and not just the rule from a ready model. The generalisation of the operation, based on analysis and deduction, appears as a need for the teacher's teaching activity as a way to give meaning to the concept for students, avoiding possible confusion in solving problems or tasks.

> *Vitor:* I think this problem helps a lot to understand this multiplication.
>
> *Sara:* But *Caderno do Aluno* addresses it a lot.
>
> *Gustavo:* We were educated like this, by the rule, without thinking much. We end up applying that which works, so there we go. And you forget to think why it's like that.
>
> *Vitor:* It is interesting that we also ask ourselves this question: "Why is it so?"
>
> (FM, 10).

This excerpt shows evidence that teachers recognise the need to understand the concept in all their relationships, not just being content with the crystallised model. The new way to understand the concept led teachers to organise the presentation of this operation to students in a way to reveal those essential relations not only manifested in the use of the crystallised rule.

> It's complicated, but once you get the hang of it, you start to understand better. Now I can talk about fractions in the classroom. Before I didn't take any chances, I went by the rule anyway. Now, I feel more confident. (Vitor, IC, 8).

The proposal presented by the group and by teacher Vitor's statement reveal the change in understanding, in the organisation of teaching and use of

official material, as it brings constitutive elements of theoretical knowledge, as described by Davidov (1988). It is worth remembering that, as we presented earlier, the teacher began the training experiment arguing that the best way to approach the multiplication of fractions was to start as "as simple as possible... numerator with numerator and denominator with denominator and multiplying as usual" (Vitor, FM, 8).

By theoretically understanding the multiplication of fractions, the use of the material gains a new dimension, becoming a mediating instrument in the teaching activity of the teacher that starts to avoid the presentation of the operation by the logic of the model from the rule. Despite recognising the possibility that the official material can be used as an important mediating instrument, teachers also understood that it could not be the only instrument, since, in addressing some concepts, the material does not bring all the necessary elements for the complete development of theoretical thinking.

By understanding the official material as a possible mediator, teachers began to understand it as a means in the process of organising teaching and not as an end in itself. Thus, they demonstrated an understanding that the official material could not be used as a mandatory manual to be followed in classes, as reported by teacher Isabela on how she interpreted the use of official materials, considering the guidelines given by the management team, justified by the command given by the central bodies: "we are practically obliged to use the *Cadernos*" (Isabela, FM, 10), but rather, that this material should be used from a critical and conscious analysis and coherently with the teacher's intentionality.

After the experiment, the teachers gave indications of intentionality and autonomy when working the learning situation produced or the task chosen in the official material, emphasising the need for students to understand the operation in all relationships and not just the crystallised rule. Thus, data analysis shows that the development of theoretical thinking provides a change in the attribution of meaning to crystallised models and rules, valuing the relationships of the concept beyond appearance. Consequently, it also changes the meaning that the teacher attributes to the mediating material in the teaching activity and, in particular, changes the way of choosing, using or adapting the official material as a mediating instrument in the teaching activity.

## CONCLUSIONS

Throughout this article we aimed to present research results that demonstrate the relationship between the development of teachers' theoretical thinking on multiplication of fractions and the relationship that teachers attribute to mediating instruments in teaching activity, in particular, the way teachers choose, use, or adapt the official materials of the state of São Paulo in teaching organisation.

For this, we presented the theoretical references from the historical-cultural perspective that supported the research and, particularly, elements of the activity theory (Leontiev, 1978, 1988) and the developmental teaching theory (Davidov, 1988). Consistent with these references, we presented a formative experiment taken as a methodological strategy for research developed with mathematics teachers of elementary school of the state of São Paulo.

The data analysed revealed that initially, the teachers presented operations with fractions involving mechanised rules and repetition of exercises, without revealing the essence of these operations. However, as they theoretically understand the meaning of the operation, especially for this article, the multiplication operation between fractions, they begin to reveal through their statements and teaching proposals the understanding of the importance of the interrelationship of arithmetic, algebraic, and geometric elements. Besides the teachers' indicating evidence of understanding about the importance of the interrelationship of mathematical elements, they also began to recognise the importance of putting the student in motion in the search for the solution, respecting their main activity.

At the same time, before the training experiment, teachers used the official teaching material to reproduce the models they brought, in which students classified the exercises from the models and repeated them several times, which was understood as a way to ensure the learning of the concept in those times. As teachers demonstrate evidence of theoretical thinking about the concept, the development of triggering situations starts to show signs of resignification to official teaching materials. During the appropriation of the theoretical elements of the fraction, teachers made relations and references to this material, no longer as reproductive support material of the models previously given, but as a mediating instrument in the organisation of teaching,

to develop theoretical thinking about the fraction, involving all external and internal relations.

Given these results, we can conclude that teachers realised that the official material of the state of São Paulo that accompanies their daily teaching practice can be a mediating instrument in teaching activity on the concept of fraction, rather than a manual to be followed. In this sense, the instrument assumes the expected role of mediation in the teaching process and not of an end in itself.

Based on this analysis, we identified evidence of change in the meaning attributed to official materials, which are at the service of the teacher's teaching activity and not the other way around, thus materialising as a mediating instrument of their teaching activity. According to Marx (1996), when the instrument materialises the conscious action of man for a given purpose, this instrument becomes an external mediator of the action of human labour.

We conclude that it is important to think about teacher education beyond the mere execution of public policies, which can lead to alienation of their work, but rather, think about mathematics teacher education seeking to develop a conscious, autonomous, and creative activity. For this, it is important to offer the teacher the knowledge of the entire historical and cultural construction of the mathematical concept, bringing them closer to the development of theoretical thinking. From this theoretical understanding, we expect that teachers, intentionally and consciously, select the best mediating instruments of their teaching activity.

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## AUTHORSHIP CONTRIBUTION STATEMENT

V.D.M. was responsible for the guidance on the theoretical assumptions and methodological referrals. I.O.R. collected and analysed the

data in light of the theory and methodology chosen. Both authors discussed and analysed the results and contributed to the writing of this article.

## DATA AVAILABILITY STATEMENT

The data of this research will be made available by the authors upon reasonable request.

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