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A Sequence of Activities for Teaching Diophantine Equations: Possibility to Expand the Knowledge Base of Future Mathematics Teachers

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ABSTRACT

Context: One of the challenges in the initial of mathematics teachers is to articulate the contents studied in the disciplines at the university with the themes of Basic Education. **Objectives**: to analyse, in the light of the assumptions established by Ball, Thames, and Phelps, the knowledge base for the teaching of Diophantine Equations of a group of future mathematics teachers participating in an experience of a teaching sequence. **Design**: the principles of the Design Experiments methodology were observed. **Environment and participants**: The study involved a group of ten students of the Mathematics Teaching Degree Course of a campus of the Federal University of Sergipe. **Data collection and analysis**: We analysed the reflections of the group of pre-service teachers, written records and audio recordings, generated by the experience of a teaching sequence on Diophantine Equations, which explored the relationship between a content of the Theory of Numbers and themes that would be taught in Basic Education. **Results**: The results show that the experience of the sequence provided the group of pre-service teachers with reflections on the difficulties that Basic Education students may have in situations involving equations and on how teachers could help students overcome them. **Conclusions**: Throughout the process, we observed advances in the understanding of the mathematical object, expanding the teaching professional knowledge base of the participants on the subject. This study concluded that a broader focus on the discussions to be promoted in the initial teacher education is needed to include and emphasise the relationship between the contents seen in the specific disciplines of Mathematics and those that will be taught in Basic Education.

Key words: Initial Education; Teaching Professional Knowledge; Teaching Sequence; Diophantine Equations.

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Uma Sequência de Atividades para o Ensino de Equações Diofantinas: Possibilidade para Ampliar a Base de Conhecimentos de Futuros Professores de Matemática

RESUMO

Contexto: Um dos desafios na formação de professores de matemática é fazer a articulação entre os conteúdos estudados nas disciplinas na universidade e os temas da Educação Básica. **Objetivos**: analisar, à luz dos pressupostos estabelecidos por Ball, Thames e Phelps, a base de conhecimentos para o ensino de Equações Diofantinas de um grupo de futuros professores de Matemática, participantes de uma formação. **Design**: observou-se os princípios da metodologia *Design Experiments*. **Ambiente e participantes**: O estudo envolveu um grupo de dez estudantes do curso de Licenciatura em Matemática de um campus da Universidade Federal de Sergipe. **Coleta e análise de dados**: Analisaram-se as reflexões do grupo de licenciandos, registros escritos e gravações de áudio, geradas pela vivência de uma sequência de ensino sobre Equações Diofantinas, que explorou a relação entre um conteúdo da Teoria dos Números e temas que seriam ensinados na Educação Básica. **Resultados**: Os resultados apontam que a vivência da sequência propiciou ao grupo de licenciandos reflexões sobre as dificuldades que estudantes da Educação Básica podem ter em situações envolvendo equações e sobre a forma como poderiam auxiliar seus alunos a superá-las. **Conclusões**: Ao longo do processo observaram-se avanços na compreensão do objeto matemático, ampliando a base de conhecimentos profissionais dos participantes sobre o tema. Este estudo mostrou a necessidade de um enfoque mais amplo nas discussões a serem promovidas na formação inicial de professores, para incluir e enfatizar a relação entre os conteúdos vistos nas disciplinas específicas de Matemática e aqueles que serão ensinados na Educação Básica.

Palavras-chave: Formação Inicial; Conhecimento Profissional Docente; Ensino de Sequências; Equações Diofantinas.

INTRODUCTION

With this article, we aim to present and discuss the data of research conducted within a discipline of the Mathematics Teaching Degree of a public university, in the light of the assumptions established by Ball, Thames, and Phelps (2008), to study the knowledge base for the teaching of Diophantine Equations of a group of preservice mathematics teachers. By introducing a sequence on Diophantine Equations to preservice teachers, we sought to show a direct link between those equations – an integral part of the content of disciplines that address Number Theory, studied at the university, and themes proposed in Basic Education – to expand the group's knowledge to teach the topic. We also pursued to discuss a way to develop skills in elementary school students so that they can build arguments and demonstrations on the subject, without necessarily reaching the formal proof.

To this end, we begin by introducing a series of problems that lead to a single equation with more than one unknown – equations that in the real domain would be classified as indeterminate and that in the whole domain can have a finite number of solutions in the context of the problem studied. We proposed that the students investigated equations of that nature in the set of positive integers, using tables and in contexts close to real situations, and related them to the context adopted in the scope of Number Theory.

We aimed to reveal to the pre-service teachers an approach that evidenced guiding principles of curricula that highlight the issue of contextualisation of content - not only in the context of everyday life and other areas of knowledge, but also in the universe of cultural elements inside Mathematics. Our choice of Diophantine Equations is justified because it constitutes a context for

> investigating and establishing conjectures about different concepts and mathematical properties, employing strategies and resources such as observation of patterns, experiments and different technologies, identifying the need, or not, for an increasingly formal demonstration in the validation of those conjectures. (Brazil, 2018, p. 540)

This competence confirms the importance of including in the initial teacher education a work aimed at the development of skills related to the construction of arguments and proofs with their future students. As we share those principles, to elaborate the activities that make up the learning situations, we base this work on materials that are aligned with that proposal: *Caderno do Professor* (São Paulo, 2009a) and *Caderno do Aluno* (São Paulo, 2009b), and *Experiências matemáticas* (São Paulo, 1995).

In a literature review, we did not find many recent studies articulating the teaching of Diophantine Equations to the initial education of Mathematics teachers. Among the works that approach our topic, we mention Oliveira's (2010) and Cade and Paiva's (2018). However, our investigation differs from those studies not only in terms of methodology, but, above all, by the emphasis we gave to the issue of proofs, because we reiterate that those equations constitute a very rich context to explore and discuss with future teachers the possibility of developing this more formal aspect of mathematics in high school.

THEORETICAL FOUNDATION

In 1986, Shulman discussed in an article that in the United States public policies and research on the knowledge base necessary for teaching did not focus on questions about the origin of teachers' explanations to teach specific content of the disciplines, nor on the sources of analogies, metaphors, examples, demonstrations, and reinterpretations of the teacher related to teaching objects. Nor did they discuss the training given to teachers to favour the connection between the various themes of the discipline itself. There were no reflections on the commitment of pedagogical strategies when the teacher did not present the skills related to specific contents.

The lack of concern with those issues in teacher education led to a significant "loss of emphasis" on the specific contents of the subject to be taught. This "abandonment" was called by Shulman (1986, p. 7) a "lost paradigm" and, to rescue it, this educator proposed to build a knowledge base, primarily, from the specific knowledge of the discipline – in our case, Mathematics. The categories of knowledge of the content necessary for teaching, established by Shulman (1986) to compose this knowledge base, are the Specfic Content Knowledge, the Pedagogical Content Knowledge and the Curricular Content Knowledge.

Recognising that the theoretical advances on the concepts that Shulman had enunciated in 1986 were relatively small for the teaching of Mathematics, researcher Deborah Ball and her collaborators – Michigan Group - began to work with the concept of "Mathematical Knowledge for Teaching." They considered questions such as: How are ideas of pedagogical content knowledge incorporated into practice? What have we learned from this concept and what still needs to be developed from it?

To answer those questions, Ball, Thames, and Phelps (2008) based their study on research on the types of knowledge and the observation of mathematics teaching practice, and subdivided the Specific Content Knowledge into Common Content Knowledge, Specialized Content Knowledge and Horizon Content Knowledge; and the Pedagogical Content Knowledge into Knowledge of Content and Students, Knowledge of Content and Teaching, and Knowledge of Content and Curriculum, which will be defined and exemplified according to the themes that are part of our study.

The Common Content Knowledge refers to knowledge of the mathematical content of a given theme that is not specific to those who practice teaching, but all those who have learned it should have it. That is, Common Content Knowledge is inherent to the citizen. It involves, for example, performing correct calculations and correctly solving mathematical problems. It is teachers' necessary, but not exclusive, knowledge, and teachers are expected to be at least able to know and know how to use all the concepts and mathematical procedures relevant to that knowledge, provided for in the students' curriculum.

An example of the Common Content Knowledge referring to the Diophantine Equations is the knowledge necessary to solve through personal strategies a problem such as: "To group 13 buses in rows of 3 or 5 in a garage, how many rows can be formed of each type, even though we do not know that the equation that translates this problem is called Diophantine Equation?" Theorems and all properties related to those equations would not be part of this knowledge.

Ball, Thames, and Phelps (2008) consider that the ability to study and apply mathematics is not enough, in general, for the teaching of mathematics, and consider that the Specialized Content Knowledge is important. They argue that, in teacher training, one should take into account "how" teachers need to know the contents to be taught and "how" and "when" teachers need to use this knowledge in practice.

The Specialised Content Knowledge for the mathematics teacher differs both from the necessary knowledge for the mathematician – the one who produces new knowledge in mathematics – and from that of the ordinary citizen, as it is related to the teaching activity. This knowledge involves, for example, analysing errors and what facilitates or hinders a proposed task; using "non-standard" approaches that can work in given circumstances; explaining procedures and "why" the algorithms and techniques taught work.

About the Specialised Content Knowledge of the teachers who teach or will teach Diophantine Equations, it is crucial, for example, that they know how to define and identify these equations, enunciate and interpret the theorem that makes it possible to affirm when a Diophantine Equation has a solution or not and obtain the general resolution, if any. Also, the contents related to the topic should be part of the Specialised Content Knowledge, even if they are not directly part of the content the teacher will teach, but which they will use to facilitate their argumentation and favour the students' understanding. In the case of Diophantine Equations, this knowledge would contribute to the teacher's knowing arguments involving prime numbers, divisibility criteria and Euclid algorithm, for example.

The Michigan group also considers, when referring to Specific Content Knowledge, the Horizon Content Knowledge, which is necessary for the teacher so that they can highlight the critical points of the content, make connections, guide their students to advance from their own conjectures, preserving mathematical principles, and familiarise students with the language and structure proper to the discipline.

In the sequence for the teaching of Diophantine Equations, we can identify that, from our point of view, the proposal allows the teacher to work the organisation of numerical information in tables, generalisations of regularities and investigations of properties of multiples and divisors through problem-solving.

Pedagogical Content Knowledge, according to Ball, Thames, and Phelps (2008), implies Knowledge of the Content and Students and takes into account that the understanding of the Mathematics of these students is related to their experiences, which enables teachers to predict and interpret mistakes and search for strategies to overcome them. For example, proposing to pre-service teachers to analyse productions elaborated by students about problem-solving involves Knowledge of the Content and Students within the scope of initial teacher education, since, at this time, the pre-service teacher can analyse the causes of errors and suggest actions for students to overcome their difficulties.

Regarding Knowledge of Content and Teaching, Ball, Thames, and Phelps (2008) refer to the fact that the work of teaching requires the teacher to select, organise and elaborate activities. This knowledge implies analysing the advantages and disadvantages of approaches and representations and different methods and procedures.

According to Campos and Pietropaolo (2013, p. 65, emphasis in the original),

> complementing *pedagogical knowledge*, the *knowledge of content and teaching* combines the understanding of specific mathematics contents with the understanding of pedagogical issues that can interfere in the teaching and learning process. It concerns the ability to organise instruction, the assessment of the advantages of using certain representations and examples, and the decision and choice of referrals to the approach of a content.

From this perspective, the teacher can organise a teaching sequence where problem-solving is present, starting from experiments of conjectures, passing through examples that justify the need to move forward to validate results. Therefore, the teacher needs not only the Specialised Content Knowledge and the Knowledge of Content and Students, but also knowledge to elaborate situations and choose didactic materials.

To complete the knowledge base for teaching, Ball, Thames, and Phelps (2008) present the Knowledge of the Content and Curriculum, which teachers need to articulate the content taught with the curriculum contents of previous and subsequent grades or those studied simultaneously in other disciplines. This category includes knowledge of the curricular guidelines and recommendations for the introduction and development of content.

Those categories proposed by Ball, Thames, and Phelps (2008) supported the initial conception of our project and guided the analysis of the data collected in the field research.

METHODOLOGY AND METHODOLOGICAL PROCEDURES

In the development of this study, we adopted some assumptions of the *Design Experiments* methodology from the perspective of Cobb, Confrey, Disessa, Lehrer, and Schauble (2003). According to those assumptions, in the prospective phase, we searched for elements for the conception and realisation of an approach of notions related to Diophantine Equations, to investigate the knowledge of pre-service teachers about the proposition, for classes of Basic Education, of problems whose solutions involve the resolution of these equations.

In this work, we dealt with the relationship of Number Theory with the contents related to the axis of Numbers, currently provided for by the *National Common Curricular Base* (Brasil, 2018). We suggested learning situations that favoured discussions of ideas contained in the Numbers axis, related to the meanings of Divisibility, the Greatest Common Divisor (GCD) and the Least Common Multiple (LCM). The Number Theory provided us with tools to be adapted for the analysis of different strategies for teaching these concepts and/or procedures: problem-solving involving Diophantine Equations and their meanings; discussion of the importance of teaching Diophantine Equations in Basic Education; proof of Theorems to obtain solutions to Diophantine Equations were some of the themes proposed to foster the discussion on the importance of establishing relationships between what is learned at the university and what is taught in Basic Education.

We chose the *Design Experiments* methodology because it has a dual purpose – teaching and research methodology - and allows for research in the context of construction and/or development of knowledge, which is of interest to us. Continuous assessment of partial results in *Design* allows

> the necessary reformulations to the project, in the course of the experiment, until all points that eventually constitute obstacles or misconceptions of the content being explored are treated. An initial version of the project, not completely defined, is elaborated, which is reviewed and improved throughout the experiment, depending on the results that are being observed. (Corbo, 2012, p. 199)

This cyclical process of elaboration, analysis, review, and reinvention of activities proposed as the reflective phase of *Design* and adopted throughout the experiment, resulted in the “creation” of a small “theory” - small, because it is restricted to the theme: Diophantine Equations. This “theory”, i.e., this sequence, is explained and justified here.

The experience with the sequence lasted four weeks, with two weekly meetings of two hours each, and counted with the participation of ten students of the Mathematics Teaching Degree course[1]. The participants are referred to as pre-service teacher (A), (B), ... (J) to safeguard their identities.

The learning situations were selected, elaborated and organised in a way that provided future mathematics teachers with a space for discussion and reflection on the establishment of links between different approaches to this content.

In proposing the situations, the discussions among the participants of each team allowed them to reflect on the activities experienced and gave them later the opportunity to look for solutions to the difficulties with the members of the other teams. Although the discussions were held within each team and in the big group, the partial reports were prepared individually at the end of each meeting. Besides, each student delivered a report at the end of each sequence. This data set allowed us to analyse from different points of view, as foreseen in the retrospective phase of the *Design*.

From our perspective, the analysis of the data obtained through the written records and audio recordings supported the understanding of fundamental ideas related to the teaching and learning processes of the mathematical notions and procedures involved in this sequence, and are therefore amenable to be applied again to other groups of teachers and preservice teachers.

Finally, another reason that we consider essential to explain our choice for this methodology concerns the theoretical foundation that supported our study regarding teacher education. As previously mentioned, the categories of knowledge necessary for teaching, proposed by Ball, Thames, and Phelps (2008), served as a guide for the

[1] This research was approved through ethical evaluation by the CEP/CONEP system.

design of the initial project and for the changes that proved necessary in the course of our investigation. Also, they were parameters for us to assess the partial and final results of our experiment.

LEARNING SITUATIONS AND SEQUENCE OF ACTIVITIES

In this work, each activity the research proposed aims to foster discussions and promote (re)meanings on the topics addressed, thus generating a learning situation. This set of activities is what we call a Sequence of Activities.

We chose to introduce the activities that make up the sequence to proceed to the analysis of the research data subsequently.

The first learning situation was carried out over two weeks, and the first activity proposed for analysis, Activity 1, involved the planning and presentation of sequences for the teaching of topics related to Diophantine Equations, called microclasses. Each team presented class notes and a teaching plan on one of the themes: Divisibility, Least Common Multiple (LCM) and Greatest Common Divisor (GCD).

After the introductions, we asked the students to write a report individually with a critical analysis of the activities and methodologies adopted by the teams.

Supported by Ball, Thames, and Phelps (2008), we proposed this activity aiming to evaluate the knowledge the pre-service teachers showed regarding the approach to teaching, and to foster the discussion of topics that would favour the relationships between the contents of the microclasses and the Diophantine Equations, elements that are associated with Knowledge of Content and Teaching and Knowledge of Content and Curriculum.

Then, we proposed another activity, Activity 2, composed of two parts. Initially, the pre-service teachers should present a solution for each of the five problem situations involving Linear Diophantine Equations, and, soon afterwards, they should present a proposal to develop this activity with 8th-year students, the former 7th grade. With this proposal, we aimed to identify not only the Common Content Knowledge when we analyse the responses of the pre-service teachers to problem situations, but also the Horizon Content Knowledge, from the articulations they made between the equations and other contents. The situations suggested are shown in Figure 1.

Figure 1.

Problem situations involving Linear Diophantine Equations (Adapted from São Paulo, 2009a)

Situação 1: Para agrupar 13 ônibus em filas de 3 ou 5 em uma garagem, quantas filas poderão ser formadas de cada tipo?
Situação 2: Quantas quadras de vôlei e quantas quadras de basquete são necessárias para que 80 alunos joguem simultaneamente? E se fossem 77 alunos? (Dado: uma partida de basquete é disputada por 5 jogadores, e uma de vôlei, por 6).
Situação 3: Um laboratório dispõe de duas máquinas para examinar amostras de sangue. Uma delas examina 15 amostras de cada vez, enquanto a outra examina 25. Quantas vezes essas máquinas podem ser acionadas para examinar 2 000 amostras?
Situação 4: Um caixa eletrônico disponibiliza para saque apenas notas de R$ 20,00, R$ 50,00 e R$ 100,00. Se um cliente deseja sacar R$ 250,00, de quantas maneiras diferentes ele poderá receber suas notas?
Situação 5: Deseja-se adquirir um total de 100 peças dos tipos A, B e C, sendo que os preços unitários das peças são R$ 1,00, R$ 10,00 e R$ 20,00, respectivamente. Se dispomos de R$ 200,00 para a compra, quantas e quais são as possibilidades de compras que podemos fazer?

To solve the problem situations, we expected, at first, that the pre-service teachers would build tables from observing patterns and regularities and, thus, identify the solutions. However, the use of tables would become limited given the problems in which the coefficients of the equation are very large numbers, which would justify the inclusion of a discussion on the general algorithm to find solutions to a Diophantine Equation, if any. By proposing an approach to this activity in a Basic Education classroom, we believed that pre-service teachers could suggest something that, besides leading students to identify patterns and regularities, also made connections between the problem situations proposed and the idea of multiples, divisors, and greatest common divisor.

After the activity had been systematised, we offered the pre-service teachers a suggestion for approaching Activity 2, proposed in *the Teacher's Notebook* of the State of São Paulo, as shown in Figure 2, in which they should analyse and present objectives for the activity, if applied to students of the 8th year of elementary school - Activity 3.

This learning situation could contribute to our understanding of the pre-service teachers' Knowledge of Content and Teaching and the Knowledge of Content and Teaching.

Figure 2.
Proposal of the Teacher's Notebook for Activity 2 (São Paulo, 2009a, p.54)

Resolução do exemplo 2

Montaremos uma tabela que nos permita avaliar possibilidades para **v** e **b** de tal forma que se atenda à restrição 12v + 10b = 80 (na sequência, analisaremos o caso 12v + 10b = 77).

Linha	Nº de pares de times de vôlei (v)	Nº de pares de times de basquete (b)	Total de alunos (12v + 10b)
1	0	0	0
2	0	1	10
3	0	2	20
4	0	3	30
5	0	4	40
6	0	5	50
7	0	6	60
8	0	7	70
9	0	8	80
10	1	0	12
11	2	0	24
12	3	0	36
13	4	0	48
14	5	0	60
15	6	0	72
16	5	2	80

Com as nove primeiras linhas da tabela, descobrimos uma solução do problema, que é v = 0 e b = 8. Note que o padrão seguido nas nove primeiras linhas não foi continuado, porque na nona linha já se atingiu 80, que é o número de alunos da escola na primeira situação proposta no enunciado do problema. Da 10ª à 15ª linha, identificamos que não há solução quando b = 0. O padrão com b = 0 não prosseguiu para além da 15ª linha, porque na linha seguinte já ultrapassaríamos 80 alunos. Por fim, buscando combinações de resultados da última coluna cuja soma seja 80, encontraremos mais uma solução para o problema, que é v = 5 e b = 2. Esse problema apresenta, portanto, soluções do tipo (v,b), que são (0,8) e (5,2).

Dando continuidade à análise desse exemplo, é fácil perceber que não existe solução para a equação 12v + 10b = 77. Uma justificativa razoável para isso é a seguinte:

- os múltiplos de 10 terminam sempre em 0, portanto, 10b tem algarismo das unidades igual a zero;
- os múltiplos de 12 terminam em 0, 2, 4, 6 ou 8, portanto, 12v termina em algarismo das unidades igual a um desses números;
- decorre dos itens anteriores que a soma 12v + 10b termina em 0, 2, 4, 6 ou 8 e, como 77 tem algarismos das unidades igual a 7, 12v + 10b nunca será igual a 77.

Pode-se demonstrar que:

Uma equação diofantina ax + by = c tem solução inteira se, e somente se, o máximo divisor comum entre **a** e **b** for um número que divide **c**.

O teorema que acabamos de enunciar garante a existência de soluções inteiras (inclui os negativos). Lembramos que nos cinco exemplos que estamos analisando, nos interessam as soluções inteiras positivas. Ou seja, sua aplicação em problemas desse tipo exige que se faça uma análise com critério, porque pode ser que a equação tenha uma solução com inteiros negativos e, nesse caso, essa solução não interessaria para o problema em questão.

Subsequently, we proposed Activity 4, which consists of demonstrating the theorem of existence of entire solutions of a Diophantine Equation and the algorithm to search for these solutions, if any. We chose to discuss the formal proof with the pre-service teachers, to share the ideas defended by Garnica (1996): in teacher education courses, all nuances of the theme – critical and technical reading - should be exposed. This author argues that the rigorous proof should be implemented in teacher education, not only in disciplines with specific content, but also in pedagogical disciplines, since this is an issue that involves scientific practice and could influence pedagogical practice.

We also consider, agreeing with Garnica (1995), that the teachers must know more than what they will teach – knowledge that this researcher calls "supplementary stock." To expand the repertoire of arguments and adopt more appropriate approaches for students, we chose to discuss with the group the formal proof of those results. From this activity, we also discussed that we cannot always reach the formal proof of a theorem with Basic Education students, but we can offer activities that allow them to raise conjectures and look for generalisations from observing patterns.

Those aspects led us to discuss with the group of pre-service teachers issues of the Specialised Content Knowledge, since we argue that the formal demonstration of the theorem of existence of entire solutions of a Diophantine Equation and the algorithm to search for those solutions is not necessarily part of those contents the teacher will teach. However, this knowledge will facilitate their argumentation to favour students' understanding.

At the end of the work with the four activities described here, students were asked to propose a teaching plan directed to students of Basic Education for the teaching of Diophantine Equations, Activity 5, to identify their choices and justifications for activities that favour students to conjecture, confirm a hypothesis and generalise from observing the regularities and patterns.

When preparing the Sequence of Activities, despite our intention to identify in each of the activities proposed some types of knowledge, in all learning situations, we could recognise elements of the different categories of the knowledge base proposed by Ball, Thames, and Phelps (2008). Besides, the pre-service teachers took as reference their experiences as Basic Education students and trainees in the compulsory subjects of the Teaching Degree to infer about their attitudes in the situations they had been proposed and expose their knowledge of the content and students.

ANALYSIS OF THE LEARNING SITUATIONS

This section reports the analysis we made of the discussions during the education process. We took into account the pairwise grouping of the categories of Ball, Thames, and Phelps (2008), as we understand that this analysis would be more promising due to the complementarity between the chosen pairs, namely: Common Content Knowledge and Specialised Content Knowledge; Knowledge of Content and Students and Knowledge of Content and Teaching; Horizon Content Knowledge and Knowledge of the Content and Curriculum.

Common Content Knowledge and Specialised Content Knowledge

When trying to solve the problem situations related to Diophantine Equations, the pre-service teachers did not bother, at least not at first, to explain their solutions to the Basic Education students. By analysing the extent to which they used the contents of the microclasses in solving the issues proposed in the activity, the pre-service teachers presented their strategies, as we can see in the following excerpts:

> I used some divisibility criteria and also did it by attempts, as it was the easiest way to solve the activity. (H)
>
> In a way, I ended up using some of the contents of the microclasses, but indirectly, I ended up using more of the trial and error, so I was trying possibilities. (A)
>
> My resolutions on the last three issues were somewhat rudimentary and somewhat "manual", but in several steps, I conjectured some patterns that were appearing using divisibility criteria. (E)
>
> I was not sure of anything, so the way to explain it to the student became unfeasible, because I had no basis to guarantee it, only assumptions, but, to me, the answers seemed satisfactory (G)

They used personal strategies, such as trial and error, for example, to find the correct answers to the problems. Although they did not know this type of equation, at least not by the name of Diophantine Equation, they did not doubt much in interpreting and describing the equation that modelled the problem. However, they presented more significant difficulties in finding all possible solutions to the problem and describing a criterion for this search.

Once the questions were resolved, the students were led to reflect on how they would approach this topic with students of the 8th grade of elementary school. We believe that this was the high point of the activity proposed, because the pre-service teachers realised that Common Content Knowledge, which allows us to present correct answers, as recommended by Ball, Thames, and Phelps (2008), is not enough for teaching. We see, in the following exemplary statements, that the proposition of this activity favoured reflection on one of the knowledge types needed by the teacher for teaching, the Common Content Knowledge.

> [...] among all the reflections, the most significant one is that I do not know how to explain some of those (situations) to students in this grade (8th grade). (I)
>
> My major difficulty was to give justifications that served as help to the students, after all, my resolution was based on attempts and, through them, I sought to restrict some solutions, but perhaps the students did not understand what I was trying to explain and, instead of helping them, hindered their learning. (F)
>
> In particular, I solved them, using more divisibility criteria, but I had much difficulty in formalising my idea in a way that was comprehensible to the target audience (8th graders), so I thought my answers and explanations were a little loose. (J)

These statements are examples that, although the pre-service teachers found valid strategies to solve the situations presented, they considered, like Ball, Thames, and Phelps (2008), that solving problems correctly is not enough to promote their teaching. Likewise, Zeichner (1993, p. 38) states that "knowing a given discipline is not, in itself, sufficient to be able to teach it." Although the preservice teachers initially attempted to solve the problems, we can observe that they did not consider this a legitimate strategy to propose to their future students. Perhaps they expected to present a more formal solution for the problems proposed, that involved a specific knowledge or procedure for that type of problem. However, we believed that they would consider this a good path to lead their future students to identify regularities, establish conjectures, test hypotheses, compare results.

The pre-service teachers could reflect on how to teach also during the systematisation of the group's conclusions. The statements of the pre-service teachers (J) and (I) ratify that there is a lack of articulation between the Common and the Specialised Content Knowledge (Ball, Thames, & Phelps, 2008):

> [...] after observing the answers, I could see that I used some concepts intuitively, without knowing that I was using them and that my line of reasoning, especially in the first two questions, was per the reasoning of the solution suggested, but I couldn't find a way to explain my solution. (J)
>
> [...] I realised that the way the resolution is proposed resembles the idea of what I tried to do, but the way it was systematised was much better for the students' understanding. (I)
>
> Activity 3 favoured the expansion of the Specialised Content Knowledge, as pre-service teachers came to know new arguments, new ways to find and represent solutions, using mathematical language suitable for teaching.
>
> [...] I did not know what Diophantine Equations were, and I also did not have the knowledge of this way of introducing the subject; therefore, the knowledge acquired was mathematical and pedagogical. (I)
>
> The activity ended up bringing a new knowledge that was the Diophantine Equations from the mathematical point of view. [...] the diophantine equations are, from the little I've seen, a great tool to help solve problems. (A)

However, I would not try to make an algorithm to see if there would be a solution, to later look for them, although the idea was of great benefit. (C)

Regarding the content Diophantine Equations, from the point of view of Ball, Thames, and Phelps (2008), we can infer that the pre-service teachers expanded both the Specialized and Common Content Knowledge, since Student I and Student A explained that, before experiencing the sequence proposed, they had never come across them, nor did they know about their teaching.

Knowledge of Content and Students and Knowledge of Content and Teaching

Regarding the analysis from the Knowledge of Content and Teaching and Knowledge of Content and Students (Ball, Thames, & Phelps, 2008), our sequence provided pre-service teachers with moments of reflection, in teams and the group, about the need to know different activities, forms of representation and material approaches, to make appropriate choices for the class for which the instruction is intended.

Thus, despite their little experience in classroom practice and dealing with students, taking as reference their experiences as Basic Education students and as trainees in the compulsory subjects of the Teaching Degree, the pre-service teachers could reflect about their attitudes in the situations they were presented.

When asked to respond to problem situations, the pre-service teachers were concerned with finding the correct answers, without taking into account the contents involved and/or the context in which they could work these situations with elementary school students.

For pre-service teacher (G), the proposal presented in Activity 3 to discuss those problem situations using the table to represent the data makes the visualisation easier and can favour the student's understanding, as we can see in his speech: "[...] the solution shown [Activity 3] is easy for the student to understand, especially when we insert it in the table, because it becomes visible to the students " (G).

This is a representative manifestation for the group, since many referred to this way of representing data as a facilitator for learning and discussing the solutions. The statements of students (J) and (I) ratify our analysis.

> In short, Activity 3 gives us a hint to try to organise the results in a table, which can make it much easier for students to visualise the problem. (I)
>
> [...] but, seeing the suggestion of the answer, I realised that the solutions are made by trial and error, which is the case of the table, which I found very efficient for the occasion, because the public is 8th-grade students, since, for them to understand, things must be tried and compared, facts that the table explores very well. (J)

The pre-service teacher (J) highlights the importance of the student doing the experiments, conjecturing, testing the hypotheses, comparing. In other words, he/she analyses the activity as potentially rich to be applied in Basic Education, taking as a criterion the question of the student constructing their own knowledge from experiments. This fact makes us infer that pre-service teacher (J) expanded his Knowledge of Content and Teaching (Ball, Thames, & Phelps, 2008).

However, some of the future teachers proved to disbelieve in the students' good performance.

> The 7th-grade (8th school year) students are not yet intellectually prepared to deal with equations of various unknowns. This activity alone does not support students enough so that they can create their own problems involving Diophantine Equations. (E)
>
> They may not master some content and, consequently, may not be able to find methods to help them solve it, and may not have the maturity and knowledge to solve them. (F)
>
> [...] the solution shown [Activity 3] is easy for the student to understand, especially when it is inserted in the table, because it becomes visible to the students. This activity is challenging for an 8th-year class if you don't tell them which tools to use. Even with the tools, in my opinion, it's still difficult. (G)

We recognise, agreeing with Manrique (2005), Pietropaolo (2002), and Pires (2002), that teachers are generally resistant to changes, as the conceptions and beliefs they built throughout their school life function as obstacles in the process of reflection on new ideas.

Horizon Content Knowledge and Knowledge of Content and Curriculum

Although the Diophantine Equations are not explicit in the curriculum, pre-service teachers (J) and (E) explain why the topic should be studied.

> [...] because it is something that we often see and apply in everyday life, I mean, students work with diophantine equations without realising it. Thus, this study will help students outside the classroom. (J)
>
> If we show the student that what they are studying now will help them with future content, it will be a natural way of motivating them. In the matter of applications, I believe it is very interesting to use those subjects in other contents of mathematics, thus reinforcing how beautiful this science is. (E)

Pre-service teacher (E) advocates contextualising contents with his own mathematics. An example of this, present in our sequence, is the contextualisation of the GCD and the Divisibility Criteria with the existence of entire solutions of a Diophantine Equation and the algorithm to find the solutions, if any. In pre-service teacher (I)'s speech, we also observed it.

> The resolution of problem 2 follows the same reasoning as the previous [problem-solving], but here he has already addressed the concept of the diophantine equation,

> and now I also know what it is. But explicitly applying the concept of LCM and GCD did not come to mind at the time of the resolution, only after it had been discussed with some friends. (I)

What we observed in the previous speech is that the preservice teachers lack Horizon Content Knowledge (Ball, Thames, & Phelps, 2008), concerning Diophantine Equations, so that they can make connections between the contents. The following pre-service teachers ratify our assertion when analysing their resolution of problem situations:

> Regarding the use of the LCM and the GCD, I did not use them, and I believe that no one has used them, since their use was not clear. (C)
>
> It was necessary to use the Divisibility and Factoring Criteria, but indirectly, because it was not explicitly given, it was not mentioned that to perform the exercise it would be necessary to use the contents, but I used it. (F)
>
> [...] But it took me a long time, and I couldn't answer them all. And that's why my first impression was that some of them were not at a suitable level for a fifth grade, for example. When the proposal for a resolution was presented and the content to be applied became evident, I ended up changing my opinion [...]. (A)

The pre-service teachers generally bring with them experiences of solving the exercises of some specific content immediately after the teacher has explained it. Thus, it is not usual that they decide which content to use to solve a problem, but to apply a set of concepts and properties from the last content studied. For example, we did not find any pre-service teachers' solution attempts that mobilised the study of the graph of the linear function generated by the Diophantine Equation. This may be because only mathematical theory and its immediate applications are presented, without the analysis of other suggestions of resolution.

The following statements show that the pre-service teachers have expanded their knowledge from the curricular point of view and the connections between the contents, which can provide their future students with a rich discussion involving Diophantine Equations.

> From a mathematical point of view, I learned to relate some subjects better and see how much knowledge of a subject can facilitate the resolution of a given issue that apparently does not need such a subject. (J)
>
> [...] through them it is possible for the student to develop a greater logical reasoning, besides allowing him/her to reflect and withdraw his/her own ways of solving some equations by resuming content already seen by him/her. (F)

By relating content, the pre-service teachers reinforce the idea of linking the content studied with previous ones, but little emphasis is placed on the importance of emphasising critical points for the study of future topics. For pre-service teacher (E): "We must be attentive when addressing this topic in elementary school. I believe it is no more important than its predecessors, the LCM, the GCD the divisibility, however, applying knowledge acquired by students in new content is very interesting." In another excerpt, the same pre-service teacher cites the later contents, but does not articulate them with the current subject, but takes them as motivators for its study: "From a pedagogical point of view, it shows a path to be followed thinking in the sense of using newly passed contents and motivating others."

In his speech, pre-service teacher (E) recognises the need for the horizon and curriculum content knowledge (Ball, Thames, & Phelps, 2008). Zeichner (2003, p. 47) also argues that "educators need to know their discipline and know how to transform it, to connect it to what students already know, so as to promote more understanding." We verified that the pre-service teachers did not reflect on strategies to teach students who do not have the previous knowledge required for the learning. In other words, the group is not yet concerned with how to make students advance regardless of their mastering all the mathematics of the previous grades.

CONCLUSIONS

In our education process, we interpreted records, dialogues and reflections of preservice teachers, participants of the research, about the sequence of activities for teaching Diophantine Equations in Basic Education, using ideas defended by researchers Ball, Thames, and Phelps (2008) about teaching knowledge. This choice allowed us to build the sequence and further analyse the data as a result of the conditions of this process, whose assumptions were to favour pre-service teachers' thoughts and interactions with each other.

We reiterate that the preservice mathematics teachers did expand their knowledge base, according to the categories defined by those researchers regarding Diophantine Equations, especially about Specialized Content Knowledge, Knowledge of Content and Teaching and Knowledge of Content and Curriculum.

The reflections we made throughout this research led us to conjecture that the study of the Diophantine Equations should gain special attention in the courses of Mathematics Teaching Degree, both in the disciplines of the teaching area and in those of specific knowledge.

The bias of proposing problems involving this topic, from conjectures and experimentations, should be privileged not only in pedagogical disciplines. In other words, the disciplines of mathematical content would not, in our view, be exempt from providing, as a means for the learning of the future teachers, the articulation between mathematical theory and the applications pertinent to the context of Basic Education. It is necessary to ensure that during their university course pre-service teachers have experiences close to those they will have in their teaching practice.

Likewise, the preservice teacher would also need to live situations that would allow them to feel the difficulties experienced by students when they begin a work involving equations. This would allow them to perceive and understand those difficulties and reflect on strategies that could help their students to face and overcome them.

We believe that given the importance of argumentation and proof for the understanding and expansion of content, its study cannot be linked to a single aspect that privileges the specific knowledge, at the risk of causing students to elaborate a conception devoid of meaning. In other words, the approach to this theme must consider all the nuances, all the complexity inherent to the construction of that knowledge.

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AUTHORSHIP CONTRIBUTION STATEMENT

Authors M. E. A and R. C. P were responsible for the elaboration and application of the teaching sequence and, in partnership with authors M. E. E. L. G and A. F. G. S, discussed the results, structured and wrote the article.

AUTHOR CONTRIBUTION STATEMENT

The data supporting the results of this investigation will be made available by the corresponding author, M. E. A, upon reasonable request by email.

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