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Potentials and Articulations of Knowledge of the Mathematical Analysis for Teaching Action in High School: An Epistemic Analysis of Notions of Sets from the Perspective of the Onto-Semiotic Approach

Paulo Cesar Pereira Napar[a]
Carmen Teresa Kaiber[a]

[a] Universidade Luterana do Brasil, Programa de Pós-Graduação em Ensino de Ciências e Matemática, Canoas, RS, Brasil.

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ABSTRACT

Background: the area of mathematical analysis is generally seen as something of a complex nature that contributes little to the formation of teachers of basic education in mathematics teaching degree courses. **Objectives:** in this sense, this article presents an analysis of the potential that knowledge from the mathematical analysis, particularly in notions of sets, presents for the development of didactic-mathematic knowledge required for teaching practice in high school. **Design:** using the theoretical assumptions of the onto-semiotic approach to mathematical knowledge and instruction (OSA), with regard to the didactic and mathematical knowledge of mathematics teachers, this qualitative research presents an epistemic analysis carried out on the notions of sets in the context of the mathematical analysis, seeking to relate the objects highlighted there with relevant knowledge at the high school level. **Setting and Participants**: the investigative scenario is theoretical research, therefore without participants. **Data collection and analysis:** the data source was a teaching-degree mathematical analysis book and a high-school book with focus on sets, which were analysed against the theoretical assumptions of the OSA. **Results:** the analyses showed that the knowledge of sets can be articulated at different levels of education, especially on the use of mathematical proofs and different languages, highlighting the potential for their contextualisation in high school mathematics teachers' practice. **Conclusions:** with this study it is possible to highlight the importance of analysis in the constitution of didactic and mathematical knowledge, as well as its importance for high school mathematics teachers' practice.

Keywords: Mathematical Analysis; Teaching Practice in High School; Sets; Onto-semiotic Approach; Didactic-Mathematical Knowledge.

Corresponding author: Paulo Cesar Pereira Napar. Email: paulonapar@gmail.com

Potencialidades e articulações de conhecimentos da Análise Matemática para a ação docente no Ensino Médio: uma análise epistêmica de Noções de Conjuntos sob a perspectiva do Enfoque Ontossemiótico

RESUMO

Contexto: a área de Análise Matemática é geralmente vista como algo de natureza complexa que em cursos de Licenciatura em Matemática pouco contribui na formação de docentes da educação básica. **Objetivos:** nesse sentido, este artigo apresenta uma análise sobre potencialidades que conhecimentos advindos da Análise Matemática, particularmente em Noções de Conjuntos, apresentam para o desenvolvimento de conhecimentos didático-matemáticos requeridos para a prática docente no Ensino Médio. **Design:** utilizando-se dos pressupostos teóricos do Enfoque Ontossemiótico do Conhecimento e da Instrução Matemática (EOS), no que se refere aos Conhecimentos Didático-Matemáticos de professores de Matemática, a investigação, de natureza qualitativa, apresenta uma análise epistêmica realizada sobre as Noções de Conjuntos no contexto da Análise Matemática, buscando relacionar os objetos ali destacados com conhecimentos pertinentes ao nível do Ensino Médio. **Cenário e Participantes**: o cenário investigativo é de pesquisa teórica, portanto sem participantes. **Coleta e análise dos dados:** a fonte dos dados utilizados foram um livro de Análise Matemática para Licenciatura e um livro do Ensino Médio, no que se referem ao tema de Conjuntos, os quais foram analisados frente aos pressupostos teóricos do EOS. **Resultados:** as análises apontaram que existem articulações que podem ser realizadas com o conhecimento de Conjuntos nos diferentes níveis de ensino, principalmente sobre a utilização de provas matemáticas e de diferentes linguagens, destacando potencialidades para a sua contextualização na prática de professores de Matemática do Ensino Médio. **Conclusões:** com este estudo é possível destacar a importância da Análise na constituição de conhecimentos didático-matemáticos, bem como sua importância para a prática de professores de Matemática que atuam no Ensino Médio.

Palavras-chave: Análise Matemática; Prática Docente no Ensino Médio; Conjuntos; Enfoque Ontossemiótico; Conhecimentos Didático-Matemáticos.

INTRODUCTION

Mathematical knowledge has a solid set of formal bases which have emerged and been systematised with the long history of mathematics practices. Arithmetic, geometry, and algebra are shown as conceptual pillars of mathematics, in which the union and interweaving of these fields move the sharing of axioms, definitions, theorems, and tools that serve to produce mathematical knowledge and solve problems arising from different areas. Emerging from these assumptions, outlines and fields of study appear, focused on elements that justify, demonstrate or prove objects of exploration of mathematics, as proposed by the area of mathematical analysis.

Reis (2001) highlights that mathematical analysis is an area that requires a very articulated range of arithmetic, geometric, and algebraic knowledge, considering that it establishes a high rigour deepening on the study of calculus and the set of real numbers. In the context of undergraduate courses involving the training of mathematicians and mathematics teachers in Brazil, this area is appropriately addressed by the mathematics research degree (*bacharelado*) and the teaching degree (*licenciatura*) courses, as presented in Legal Opinion 1.302 of 2001, being known for its structuring that aims at the mathematically accurate treatment of proofs and demonstrations.

Discussions about this systematisation of the knowledge of the analysis generate conflicts in the context of how these should be addressed, or not, in the mathematics teaching degree courses. This impasse appears, based on Otero-Garcia, Baroni, and Martines (2013) and Napar (2018), at the moment when we think about the professional focus of the mathematics teacher education courses: they qualify professionals to work with mathematics teaching in the final years of elementary school and high school.

Authors such as Reis (2001), Batarce (2003), Moreira, Cury, and Vianna (2005), Bolognezi (2006), Otero-Garcia, Baroni and Martines (2013) have indicated that if, on the one hand, the high mathematical rigour contained in components of the mathematical analysis should be worked with the undergraduates, on the other hand, one should think about how to approach this knowledge in the context of mathematics teachers' education. This understanding is linked to the view that formal requirements of mathematical knowledge, as conducted in the analysis, may, in a certain way, seem out of context in relation to the teaching practices of the undergraduates, leading them to question the validation and why the component is present in the teaching degree.

Taking into account this scenario, we developed an investigation within the framework of a master's research entitled *A Análise Matemática na Constituição de Conhecimentos para a atuação do Professor de Matemática no Ensino Médio: uma análise na perspectiva epistêmica do Enfoque Ontossemiótico* (The Mathematical Analysis in the Constitution of Knowledge for the Performance of the Professor of Mathematics in High School: An Analysis from the Epistemic Perspective of the Onto-Semiotic Approach). Qualitative, this research was guided by the objective of investigating articulations between the institutional mathematical knowledge of mathematical analysis for a mathematics teaching degree and those of high school that have the potential to support the knowledge of the mathematics teacher to work in high school.

Starting from an excerpt from the research mentioned above, this article presents an analysis that aims to discuss and reflect on the potential that knowledge from the mathematical analysis, particularly in the notions of sets, presents for the development of didactic-mathematical knowledge (DMK) (Godino, Giacomone, Batanero, & Font, 2017) required for teaching practice in high school. This discussion starts from a study on the chapter "Números Reais - Parte I" (Real Numbers), in the context of the notions of sets of Avila's (2006) book *Análise Matemática para Licenciatura* (Mathematical Analysis for the Teaching Degree), relating it to relevant knowledge at the high school level, based on Dante (2013). Such teaching materials are taken for analysis because they constitute knowledge references for both the mathematics teaching degree and the high school.

To conduct this discussion, we consider the theoretical assumptions of the onto-semiotic approach to mathematical knowledge and instruction (OSA) (Godino, Batanero, & Font, 2008), focusing on the epistemic dimension of didactic suitability (Godino, 2009, 2013; Godino; Batanero & Font, 2008), also relying on the notion of the didactic and mathematical knowledge of mathematics teachers (Pino-Fan & Godino, 2015; Godino *et. al*, 2017). From this theoretical approach, the evaluation criteria of the didactic analysis tool: epistemic dimension - DATED (Godino, 2009; Napar, 2018) are highlighted, which are used to analyse and propose the relationships between the knowledge emphasised.

The following presents theoretical elements that support this work: the didactic suitability of the onto-semiotic approach to mathematical knowledge and instruction.

THE DIDACTIC SUITABILITY IN THE SCOPE OF THE ONTO-SEMIOTIC APPROACH TO MATHEMATICAL KNOWLEDGE AND INSTRUCTION

The onto-semiotic approach to mathematical knowledge and instruction (OSA) (Godino, Batanero, & Font, 2008) constitutes a theoretical approach that comprises the reflection, articulation, approximation, and integration of different theoretical models related to general didactics and the didactics of mathematics. It aims to qualify and expand knowledge about the processes of studies aimed at teaching and learning mathematics, and can, according to the authors, be used to analyse, reflect on, and guide educational proposals in the scope of mathematics education.

The basis of this theoretical framework starts from the "[...] formulation of a mathematical object ontology[1] that contemplates [a] triple aspect of mathematics such as: socially shared problem-solving activity, symbolic language, and logically organised conceptual system" (Godino *et al.*, 2008, p. 11). This assumption includes the mathematical activity of problem-solving as a system of practices that can be represented and shared, in the search for systematisation and interaction of ideas in the construction and production of the mathematical knowledge of a community[2] (Godino *et al.*, 2008).

The knowledge conducted in the scope of the OSA is currently presented in five groups: (1) Systems of Practices (operative and discursive), (2) Configuration of emerging mathematical objects and processes that intervene in mathematical practices, (3) Didactic Configurations, (4) Normative Dimension, and (5) Didactic Suitability (Godino *et al.*, 2017).

Such theoretical groups are articulated to produce elements for the enhancement of how mathematics teaching and learning can be constituted, complemented, and analysed. In this context, the first four groups establish tools to conduct didactic-explicative analyses, serving to describe how systems and relationships with mathematical objects work

[1] Considering the systematics of the theoretical approach, mathematical objects can be anything or any mathematical entity to which the subjects refer, whether of a real, imaginary, or other nature (Godino *et al.* 2008).
[2] They refer to the subjects, in a given sociocultural context, who constitute interactions and communication in favour of the production and sharing of ideas and knowledge (Godino *et al.*, 2008).

(Godino *et al.*, 2008). The fifth group, which refers to didactic suitability, is a theoretical resource to investigate, assess, and analyse study processes for an effective intervention in the classroom (Godino, 2013) and in mathematics teaching and learning processes. This group refers to a process that harmoniously integrates the articulation of six dimensions, which are presented in Table 1.

Table 1
Dimensions of Didactic Suitability (Godino *et al.*, 2017)

Dimension	Description
Epistemic	It refers to the degree of representativeness of the institutional meanings emerging from the mathematical objects prescribed in the curriculum, norms, and planning, having as reference the mathematics used in the teaching and learning processes.
Cognitive	Expresses the degree of approximation between the intended and implemented meanings and the personal meanings of individuals within the teaching and learning processes.
Interactional	It looks at the degree and modes of interaction that allow identifying the epistemic or cognitive disparities and conflicts that occur in the production of meanings in the teaching and learning processes.
Mediational	It represents the degree of adequacy of the technological and methodological resources that are used in the development of the teaching and learning processes.
Emotional	Relates to the degree of factors of interest, motivation, beliefs, traditions…, which interfere or assist in the teaching and learning processes.
Ecological	Characterised as the degree of adequacy of the teaching and learning processes in the educational environment of the school and social community and the context in which they develop.

Each dimension of the didactic suitability can serve to study a specific issue, such as a task or a lesson plan, or even more global issues, such as the curriculum proposed by an institution (Godino, *et al.* 2008). According to Godino (2013), this flexibility between specific and global issues, allows each dimension to be verified and analysed separately - making possible the particularisation of the analysis of the study process on each of the perspectives of this theoretical group- or together.

In what follows, we highlight the assumptions of the epistemic dimension of didactic suitability, since it is taken as a reference for the analysis conducted in this article. Table 2 shows the components and indicators that are used in its context. The components and indicators are organised around what is called the didactic analysis tool: epistemic dimension (DATED) and refer to a process of mathematical study. In the following table, read, as a sentence preceding each indicator shown, the expression "The study process (...)".

Table 2

Assessment criteria of the Didactic Analysis Tool: Epistemic Dimension. Translated and adapted from Godino (2013)

Components	Indicators
Problem-situations	- it displays a representative and articulated sample of situations of contextualisation, exercises, and applications. - it proposes situations of generalisation of problems (problematisation).
Language	- it uses different modes of mathematical expression (verbal, graphic, symbolic, etc.), treatment, and conversions between languages. - it has a level of language that is appropriate to the students to whom it is addressed. - it proposes situations of mathematical expression and interpretation.
Rules (definitions, propositions, procedures)	- it has the definitions and procedures clearly and correctly, being adapted to the educational level to which they are addressed. - it presents the key statements and procedures of the theme for the proposed educational level.
Arguments	- it promotes situations with which the student has to argue and justify mathematical thinking. - conducts and requests explanations, evidence, and demonstrations appropriate to the level to which they are addressed.
Relationships	- presents mathematical objects (problems, definitions, propositions, etc.) relating and connecting with each other. - promotes articulations of the various meanings of objects that intervene in mathematical practices.

The didactic analysis tool: epistemic dimension (DATED) is used in this article to evaluate the institutional relationship of mathematical objects that are placed in the mathematical analysis and the high-school mathematics books. From the analysis carried out with this tool, looking at the study process that can be managed from textbooks, we seek to highlight the epistemic conditions in which mathematical objects appear in the approach to books, pointing out the potential articulations that can be considered between the mathematical objects of the mathematical analysis and those of high school. Moreover, as this is an issue focused on the knowledge that can and should be mobilised at the institutional level by mathematics teachers, theoretical support is also taken in the modelling of the didactic and mathematical knowledge presented below.

THE ONTO-SEMIOTIC APPROACH IN THE CONTEXT OF MATHEMATICS TEACHER TRAINING: THE DIDACTIC-MATHEMATICAL KNOWLEDGE

The didactic and mathematical knowledge for the formation of mathematics teachers, in the context of the onto-semiotic approach, starts from a systematisation inspired by the theoretical dimensions that are indicated by the didactic suitability. Its central axis consists of an epistemic (institutional) and cognitive (personal) view, which is based on an anthropological approximation in which mathematics is understood as human and onto-semiotic activity (focused on the notions of object and meaning) (Godino *et al.*, 2017).

This modelling of knowledge, especially the knowledge of mathematics teachers, considers that teachers should have mathematics teaching skills for the educational level they teach and that they should be able to articulate this knowledge with their corresponding peers at higher levels of education (Godino, *et al.*, 2017). In the case of Brazilian basic education, and considering the mathematics teachers' education, this could be understood as the need for teachers to appropriate the knowledge related to the mathematical practices involved according to their professional performance in the final years of elementary school and high school (Napar, 2018). Therefore, it is understood as essential that this professional has control over: mathematical knowledge, the teaching of mathematics, the curriculum (rules, guidelines, and laws), the interaction in the teacher-student relationship and the availability of mediation that is understood to be necessary for their performance in the school setting. Likewise, considering an expansion of this knowledge, the teacher should also appropriate the knowledge corresponding to later levels, which could, for example, refer to those related to mathematical practices that emerge from the advancement in research in mathematics education, as well as the articulation of these objects with knowledge from other contexts and other areas of teaching.

In this context, the OSA assumes an idea of integration between teachers' knowledge, aiming to overcome the potential dichotomy between that knowledge that is didactic or mathematical. The vision here permeates the currents of knowledge required for practice and specific knowledge for mathematical action, meeting a possibility that considers the teaching action as a joint professional scenario, in which the teacher performs and acts with both knowledge in the different factors related to the teaching and learning processes (Napar & Kaiber, 2018).

Table 3 shows the dimensions of the didactic and mathematical knowledge that have the same designations as the dimensions of the Didactic Suitability, as well as a description of each of these dimensions in the perspective of this model (column two).

Table 3
Dimensions of the Didactic-Mathematical Knowledge (Godino *et al.*, 2017)

Dimensions	Description
Epistemic	It refers to the didactic and mathematical knowledge about its own content; about the particular way in which the mathematics teacher understands and knows mathematics and relates it to the context of the knowledge of his/her teaching practice.
Cognitive	It refers to the knowledge of how mathematics teachers learn, rationalise, and understand mathematics in the process of their learning and teaching.
Affective	It concerns the affective, emotional, aspects of attitudes and beliefs about mathematical objects in the teaching and learning process.

Dimensions	Description
Interactional	It refers to the teacher's knowledge in the teaching of mathematics: in the organisation of tasks and activities that aim to reduce the difficulties of their students on mathematical objects and on the school context; on the interactions that are established in the classroom (teacher and student).
Mediational	It is the teacher's knowledge of technological, material, and temporal resources, appropriate to enhance their students' learning.
Ecological	Teacher's knowledge of mathematical content related to other disciplines and curricular, socio-professional, political, and economic components that drive the processes of mathematics instruction.

The epistemic perspective, within the didactic-mathematical knowledge (DMK) considers a set of mathematics teachers' specialised content knowledge, which, from the different contextual factors and mathematical processes (problem situations, languages, concepts, propositions, procedures, and arguments), are articulated in two interrelated notions of mathematical knowledge of the teacher: common content knowledge and expanded content knowledge.

Godino *et al.* (2017) point out that common content knowledge refers to teacher knowledge that is shared with students at the level at which the teacher teaches mathematics, thus connecting with the mathematical objects attributed to classroom practices. The expanded content knowledge is defined as the knowledge shared within the later levels of education, as well as the relationships with other contexts and areas of knowledge, the connections with investigations in mathematics education, and the very formation of the mathematics teachers.

Based on Godino *et al.* (2017), an interpretation of the potentialities of these notions in this investigation is taken. We conceive that common content knowledge constitutes the possibilities of mathematical connections that mathematical analysis knowledge has to support specific teacher knowledge in their practices in the classroom for the levels they teach. The expanded content knowledge refers to the development of competencies to identify, signify, and relate the mathematical knowledge of mathematics analysis with those of high school, as well as other contexts that may be involved in the professional practice of mathematics teachers.

Having listed the theoretical characteristics that guide this investigation, the following section addresses the methodological aspects that are used to justify the analysis materials, as well as the constitution of this research.

METHODOLOGICAL ASPECTS

This article aims to present an analysis of the potential that knowledge from the mathematical analysis, particularly in notions of sets, offers for the development of didactic-mathematical knowledge required for teaching practice in high school, and it is around this objective that the methodological assumptions are intertwined. As already pointed out, this study starts from an analysis of the chapter “Números Reais - Parte I” (Real Numbers), in the context of the notions of sets of the book *Análise Matemática para Licenciatura*, relating it to relevant teaching practices knowledge at the high school level, based on Dante (2013). Furthermore, we adopt a common mathematical subject among the materials of analysis: notions of the use of the set theory.

We consider the books used as mathematical references, which serve as a data source for the analysis, based on two arguments. First, the use of the book *Análise Matemática para Licenciatura* is justified because, according to Napar (2018), it is a common basic bibliography among five teaching degree courses in mathematics in the metropolitan region of Porto Alegre/RS. It is also is a referent written by a renowned author in the area of mathematics with substantial attempts to direct the analysis to the teaching degree courses. Second, the use of Dante’s high-school textbook (2013) is justified because it was the textbook most chosen by teachers of basic education for the mathematics component of high school in 2015 (Napar, 2018), when the main investigation was developed.

For the analysis, we used the assessment criteria of the DATED, already presented. We start from the mathematical analysis book and evidence elements of the study based on the assumptions of the tool, meeting an articulation with the potential of the knowledge worked in activities of the high school book. We also use the epistemic notion of didactic and mathematical knowledge, seeking to point out characteristics of common and expanded content knowledge, conducting the theoretical contributions in an articulated manner with an approach to meet the objective proposed in this work. To establish relationships with different levels of education, an interrelationship of component of relationships with the other elements of the theoretical tool is also considered, thus underscoring the potential for articulations and approximations between the mathematical analysis and the high school in the context of problem-situations, rules, language, and arguments.

Based on this methodological design, the following section presents the analysis conducted and performed from the emerging theoretical assumptions of the onto-semiotic approach.

PRESENTATION AND DISCUSSION OF THE ANALYSIS

The section of the mathematical analysis book, called “Números Reais – Parte I” (Real Numbers) - excerpt in which the notions of sets are presented -, argues about mathematical objects that contribute to reflections and initial notions around the formation of real numbers. It involves, in an organised way, aspects of (1) Rational and Irrational Numbers; (2) Notions of sets (what this object is, operations, and types of representation); (3) Enumerability of sets, as well as ideas of (4) Finite and infinite sets and (5) Historical and complementary notes on the set theory. The study presented in this work seeks to discuss the elements indicated in (2) Notions of sets, considering the components and indicators of the didactic analysis tool: epistemic dimension (DATED).

In the context of this work, the analysis was grouped into: problem-situations and relationships; languages, arguments, and relationships; rules and relationships. Contrary to what DATED suggests, the analysis here is done in an interrelated way, because we realised that the way the mathematical objects of the book section was presented indicated several relationships that should be taken into account, such as: the relationships of the problem-situations of the analysis with problem-situations of the high school book; the languages and arguments that relate to the same elements at the high school level; and the rules that, despite being presented at a higher level of language, still deal with the same mathematical object. In this sense, we understood as imperative to carry out the groupings to better discuss the analysis and highlight the evidence of knowledge that meets the objective of this article. The following presents the analysis carried out.

From problem-situations and relationships

The presentation of the notions of sets begins with the idea of what would be a set that "[...] can be defined by the simple listing of its elements between keys or by the specification of a property that characterises its elements" (Ávila, 2006, p. 29). From there, it follows that, for the use of the concept of whether an element x belongs to a set A, the relationship of contained or not contained between two sets A and B, union operation (set of elements that are in at least one of the sets), intersection operation (set of elements that are in both sets) and operation of difference between sets (set of the remaining elements after removing the common elements from the first set with the second set).

Ávila (2006) explains that the approach presented is taken only to be used by the study that is being proposed in the section and, in a note, a criticism about how the notion of sets is conducted in the context of high school is stressed. The criticism pointed out by the author refers to the supposedly long treatment addressed in high school books, indicating that the notion of sets at this level of education usually refers only to the use of this object as language and notation. In this case, its use concerns only the use as auxiliary tools, and should therefore be introduced at times that are necessary for the object of study in question (Ávila, 2006).

On this issue, we agree, in part, with the author, since the idea of sets in high school transcends a simple symbolic relationship to be used in mathematical objects. For example, this mathematical object can be used as a form of reasoning called syllogism. Dante (2013) points out that syllogisms would be fundamental relations of mathematical deduction that emerge from situations involving the notion of sets. For example, based on this author, it would be feasible or relevant, at the high school level, to use situations such as:

The following sets are defined:

I: set of Canoenses (people from Canoas, RS)[3]

J: set of people from Rio Grande do Sul

K: set of Brazilians

From the information contained in the area of general knowledge about Brazil, it is possible to infer that:

(1) Every Canoense is from Rio Grande do Sul.

(2) Every citizen from Rio Grande do Sul is Brazilian.

The logical implication arising from these statements ensures that every Canoense is Brazilian. In terms of language of sets, we say that if set *I* is contained in *J*, just as *J* is contained in *K*, so *I* is contained in *K*, by deductive transitivity.

The context presented refers to an activity of potential problem-situation that can be used by teachers who teach at the high school level. It is important, in this sense, that these professionals own this knowledge and its possibilities to have them implemented in teaching practice. We understand that this approach is configured as a teaching path pertinent to the teachers' common content knowledge, being a required knowledge to be worked in the context of high school level. Likewise, the deepening of the study of sets, as pointed out in the proposals for activities in Ávila's (2006) mathematical analysis, could indicate issues of link of mathematical knowledge of the teacher's practice, corroborating a continuous work on how this study would be critical in the constitution of their knowledge. Also, as knowledge was contextualised and related, it would be bound to an expanded view of content knowledge, not only meeting the common practices of mathematical knowledge highlighted there, but also the actions deemed essential for the set of the mathematics teachers' knowledge and professional competencies to work in the classroom (Napar, 2018).

Although it is possible to have an "excessive" treatment in the notion of sets, as indicated by Ávila (2006), we realise that when well conducted, such an approach can corroborate so that the teacher can reflect on other methods and ways to teach the notion of sets, seeking to enrich their practices in the classroom. In this perspective, bringing a counterpoint to Ávila's (2006) claim that high school books bring an exhaustive approach to sets, we argue that it is not necessarily the task of the high school textbook to synthesise the use of the knowledge presented. Based on Godino *et al.* (2017), it would be up to the teacher, within the set of their competencies, to select and transpose the mathematical objects that are needed for the student's knowledge, as indicated in the curriculum.

[3] The term "Canoense" refers to the group of people whose place of birth is the municipality of Canoas, located in the metropolitan region of Porto Alegre in the state of Rio Grande do Sul, southern Brazil.

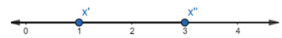
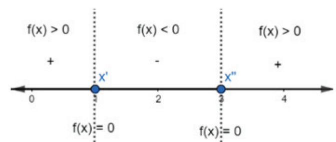
In the context of possible contextualisation problems to insert the knowledge approach in the indicated study, Ávila (2006) proposes situations of use of the mathematical object in a context inserted in mathematics (intramathematical knowledge, according to Godino *et al.*, 2008). One of the situations proposed, for example, refers to the identification of the solution set for the trinomial "$x^2 - 4x + 3 > 0$".

To find the solution of this trinomial, the author suggests that we think about the possibilities of real values for "x" in which the trinomial assumes positive values, that is, the solution set is "S = $\{x \in \mathbb{R}; x < 1\} \cup \{x \in \mathbb{R}; x > 3\}$". We identified, at first, that the resolution assumes, after all, that the reader already knows the necessary strategies to locate the solution to this inequality. In fact, because it is a book of studies within the scope of mathematical analysis, the undergraduate student is expected to have already developed, throughout the integration of the different curricular components of the degree in mathematics, a set of competencies that lead him to have sufficient knowledge to deal with this mathematics, thus, on the one hand, an approach to knowledge that is the basis of what is being presented is dispensable (Napar, 2018). However, we assume that the approach of this type of procedure, with regard to the search for the solution of inequality, could present connections of the knowledge of the analysis with didactic situations inherent to high school that are addressed by Dante (2013).

As an example, we underscore a method to find a solution to a second-degree inequality, thinking, in a high school context, through the analysis of the behaviour of the inequality from the "sign" of its intervals. Referring to Dante (2013), teachers of the first year of high school should work arithmetic and geometric procedures related to the study of the solution set of inequalities. In the case of the inequality previously highlighted, this could be worked with high school students, as shown in table 5.

Table 5
Method to find the solution to the inequality based on a study of the sign (The authors)

	Let us assume the inequality: $x^2 - 4x + 3 > 0$. **Procedures**	**Representations**
1	Inequality is considered as a function of type $f(x) = x^2 - 4x + 3$. In this case, the "zeros of the function" are located first (when f (x) = 0) by a convenient method (formula to solve second-degree equations, sum and product, factorisation, etc.). Here, we use the sum and product method, which, in this case, being "a = 1; b = - 4; c = 3", refers to considering that the sum of the roots of the equation results in "-b" and the product results in "c". We assume, in the representation: x the possibilities of roots (x' and x'')	$x = \begin{cases} x' + x'' = -(-4) \\ x'.x'' = 3 \end{cases}$
2	From the system structured in (1), we determine that the roots are $x' = 1$ and $x'' = 3$, since these values satisfy the equations of the system, and they compose the solution set *S*.	$x = \begin{cases} 1 + 3 = 4 \\ 1.3 = 3 \end{cases}$ $S = \{x', x''\} = \{1, 3\}$

	Let us assume the inequality: $x^2 - 4x + 3 > 0$. Procedures	Representations
3	The roots found in (2) are represented on a real straight line.	
4	Using random values that precede, are between and succeed the roots, we verify what happens with the sign of the function. It is known that, as x' and x'' are "zeros of the function", then at these points f(x) = 0. Likewise, the values of the function are inspected at x = 0, x = 2 and x = 4, for example.	If $x = 0$, $f(0) = 0^2 - 4.0 + 3 = 3$; If $x = 2$, $f(2) = 2^2 - 4.2 + 3 = -1$; If $x = 4$, $f(4) = 4^2 - 4.4 + 3 = 3$; If $x = 1$, $f(1) = 0$; If $x = 3$, $f(3) = 0$.
5	From the inspection used in (4), we find that: $f(x) > 0$, if $x < 1$; $f(x) < 0$, if $1 < x < 3$ $f(x) > 0$, if $x > 3$	
6	We conclude that the values of the function are positive for when $x < 1$ or $x > 3$. Similarly, we can affirm that the solution set for the initial inequality refers to the range, which can be written in algebraic notation of sets, where the values of x make f(x) positive.	$S = \{x \in R; x < 1\} \cup \{x \in R; x > 3\}$

In a way, we recognise that the use of the procedures and representations mentioned could highlight a potential for a didactic-explanatory view and that, according to the context considered, it could agree with the assumptions of the mathematical objects used there. Not only such procedures could be taken as initial objects to contextualise a more complex scenario, such as the study of the real line and the notions of continuity, which is shown as necessary knowledge for the study of functions, based on Dante (2013). To this end, the approach and relationship of this knowledge could attribute treatments to the common content knowledge, because it is a reference that, as indicated before, mathematics teachers must master to teach in high school. Still, this approach would allow for the enhancement of the relationships and problem-situations that reflect on the teaching process used and the relationships with mathematical objects, raising questions and self-reflection about what is being treated, thus listing a path of expanded content knowledge (Godino, *et al.*, 2017; Napar, 2018).

Of languages, arguments, and relationships

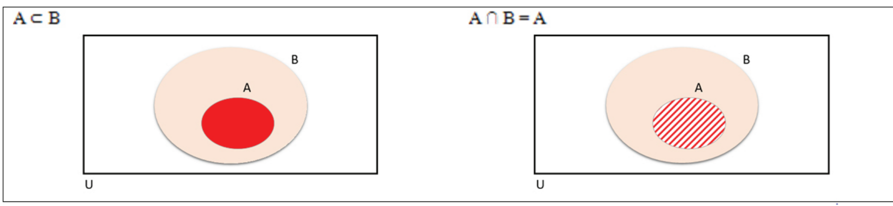
The analysis allowed us to observe that Ávila's section (2006) on sets has a wide range of problems (or exercises, are they are called in the book) that require demonstrations and mathematical proofs from those who solve them. Moreover, the book requires the use of proof arguments that, potentially, can permeate forms of algebraic, arithmetic, geometric, and natural language representation and resolution. It is also noteworthy that the various language conversions presented are requested through problem-solving, especially those that require mastery in natural and geometric language.

As an example of what we are indicating, we highlight: “Prove that $A \subset B \leftrightarrow A \cap B = A$. Make an illustrative diagram” (Ávila, 2006, p. 32).

To solve the task, a diagram is considered an illustrative, as shown in Figure 1. In the first column, there is a geometric representation for the first member of the double implication, and in the second column for the second member. In both instances, A and B are sets related by an operation (contained and intersection, respectively) and U refers to the universe set.

Figure 1

Geometric representation for the proof indicated (the authors)



Concerning the algebraic notation, the reader is informed that set A (filled in by the smallest representation in the first column) is contained in B, assuming that they are different (there is no need to verify it when A is equal to B, since the hypothesis is that “A is contained in B” and not “A is contained or is equal to B”). Then, the representation denotes visual information in which A is a set that is within set B and, therefore, every element “x” belonging to A also belongs to B. In addition, we verify, based on the second member of the double implication (column number two) that, considering the same sets, the representation of the intersection between A and B (hatched region) refers to the totality of the representation of set A. In this case, we realise that every element “x” belonging to A is a common element belonging to B and, therefore, the intersection is the set A itself.

Based on the representation used and the argumentation, the double implication sentence is ensured as valid and true for said case.

We understand that “[...] issues such as this enable the mathematics teacher to articulate potentials and different representations involving mathematical proofs that may be useful in practice as common content knowledge” (Napar, 2018, p. 133). They demand, in this sense, that the teacher thinks, studies, and organises the set of languages and arguments they must find a solution and to that problem. Moreover, they indicate a generalisation character, presenting a condition that is not solved by a particular case algorithm, but by sequencing that requires argumentation and sometimes different representations that can be constructed from different languages (Napar, 2018).

In this context, the mathematics teacher is required to have contact with a theoretical deepening that is at an adequate level of mathematical rigour, covering different mathematical perspectives that can guide him/her to develop skills to reflect on the mathematical objects he/she teaches, knowing how to analyse the mathematical justifications that he/she will request and present to his/her students, associating the notion of expanded content knowledge.

Of rules and relationships

The definitions and procedures adopted in Ávila's book section (2006) are clearly and accurately shown, highlighting the properties at the educational level targeted. Also, at first, relevant elementary knowledge is presented, such as: the set theory, set notation, pertinence and inclusion relations, definition of sets and definition between union and intersection operations.

When presenting the properties of operations between sets, equalities that are indicated as valid and true are highlighted, but the reader must prove them mathematically. They are presented in Figure 2.

Figure 2

General properties of the notions of sets, according to Ávila (2006*, p. 30)*

General properties

We will give below a series of equalities between sets, which are demonstrated by proving in each case that the first member is contained in the second and that the second is contained in the first:

$A \cup B = B \cup A$; $A \cap B = B \cap A$; $A \cup (B \cup C) = (A \cup B) \cup C$;
$A \cap (B \cap C) = (A \cap B) \cap C$; $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
$A \cap (B \cup C) = (A \cap B) \cup (A \cap B)$.

De Morgan's laws, in the case of two sets *A* and *B*, state that:

$(A \cup B)^c = A^c \cap B^c$;
$(A \cap B)^c = A^c \cup B^c$.

In other words, the complementary of the union is the intersection of the complementary and the complementary of the intersection is the union of the complementary.

We consider, based on Dante (2013), that the concepts and properties presented in Figure 2 are directly related to ideas of the high school book, since we observed that both books, *Análise Matemática* and the high school book, highlight the same characteristics as the properties of sets. Thus, it is possible to perceive that there is a relationship between the two contexts, inferring a possibility of direct connection. Furthermore, as highlighted throughout these analyses, we noticed that the book of analysis, besides declaring the properties, asks the reader/teacher to demonstrate them, considering mathematical justifications that show that they are, in fact, valid. This view denotes that there is a concern about not showing only knowledge about what the propositions, theorems, and properties are, but also that the reader/teacher makes efforts to conduct arguments that justify and demonstrate them.

In this context, it is noteworthy that the mathematical knowledge involved in these rules makes it possible to have a vision that can reference, as content, the knowledge involved in the context of the high school classroom (common content knowledge). Still, it relates to the objects that are presented at that level of education, corroborating a mathematical activity in which teachers relate the knowledge developed throughout the mathematical analysis with their mathematical practices focused on teaching (expanded content knowledge).

FINAL CONSIDERATIONS

In the research carried out, the analyses allowed us to perceive that there are relationships and articulations of the mathematical analysis and the high school textbook that can be made with mathematical objects (notions of sets) and that, also, depending on how they are conducted, they can present elements that discuss not only the knowledge produced as a mathematical reference, but also their intertwining with teaching practices. We observed that the elements related to structures and notions of sets, especially regarding problem-situations and operations with sets, are the mathematics that teachers must master conceptually, because they are expected to be able to offer their students with a vision of mathematics that is not reduced to truths assumed as unique, but as ideas that can be shown, argued, discussed, justified, and built together.

The analyses also indicated the understanding that mathematics teachers know the need to use mathematical proofs and demonstrations in high school, articulating them with different mathematical objects, variation between languages (natural, algebraic, and geometric), and logical mediation. On the one hand, the proofs and demonstrations conducted in *Análise Matemática* show the educator must be able to interpret and execute, mainly, justifications in natural language. On the other hand, the educator must know conversions to this or that language, which will allow them not only to signify and appropriate better what is being studied, but will also expand their repertoire of approaches to the different themes that are proposed in high school.

There is the understanding that the relationships pointed out in the analyses indicate important articulation movements for the mathematical expanded and common content knowledge, which would thus offer possibilities for a solid knowledge base for mathematics teachers. Moreover, those relationships go beyond mathematical support to signify the importance of studying these objects of mathematical analysis, but also configure a didactic background that constitutes didactic-mathematical knowledge for reflections on mathematical practice and the understanding of mathematical knowledge and competencies the teachers need to teach.

We also realise, based on the onto-semiotic approach, that mathematics teachers need to develop their mathematical specialised content knowledge adequately: common content knowledge (minimum knowledge to teach mathematics at the level the teacher teaches) and expanded content knowledge (knowledge related to later levels or other contexts, such as the knowledge about mathematical analysis). Highlighting the expanded knowledge, which is closely related to what is being discussed in this article, we consider that *Análise* consists of high rigour mathematical elements, developing topics with demonstrations, proofs, and mathematical justifications that are deemed necessary for the teacher to know the vision in which the essence of mathematical knowledge is shown. Just as the teachers need to master and know mathematics to teach at the educational level they teach, we recognise that they also need to know the most complex structures of knowledge to understand the nature of mathematical objects and, consequently, hold expanded knowledge about what they need to teach. However, when it comes to an analysis focused on undergraduate teaching degree courses, we realise that the assumptions worked with the undergraduates must be bound to the issues of teaching practice and the context of the classroom in which the teacher will have to teach. This understanding is given because the main focus of the teaching degree is to qualify teachers to work in basic education and, thus, approximations between what they learn in academia and what they teach in their professional setting are essential.

Based on the context mentioned, it is important that the teaching of the mathematical analysis incorporates elements of the practice of the basic education mathematics teacher, presenting a look at the logic, justification, formalisation, argumentation in the constitution of mathematical objects and ideas. Those elements would thus be corroborating the act of thinking, teaching, and learning mathematics, enabling preservice teachers to have contact with the contextualisation of this knowledge, qualifying their future teaching practices. We also believe that it is not a question of reducing the level of knowledge that is addressed in the analysis, but of listing the possibilities of the relationships that it has with the context of teachers' performance, allowing for greater development of their competencies with their expanded content knowledge.

Finally, it is noteworthy that the relationships of mathematical knowledge between *Análise Matemática* and the high school book are not always trivial. However, we highlight that proposals are needed, such as this one, in the convergence and relationship between the presentation of rigorous mathematics (which is not considered dispensable in the mathematics teacher education,) and the formative elements required for teaching practice. Thus, in the context of mathematics teaching degree courses, we recognise how important it is that analysis teaching is articulated with the perspective of qualifying teachers to work in basic education and, thus, it is paramount to think and rethink the way in which the knowledge employed there is conceived, in view of the professional action of those educators who teach mathematics.

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AUTHORSHIP CONTRIBUTION STATEMENT

P.C.P.N. and C.T.K. conceived the idea of this article. The first author collected and verified the data presented. Both authors discussed and collaborated in the structure of the analyses and the final formulation of this article.

DATA AVAILABILITY STATEMENT

The authors agree to make their data available at the reasonable request of a reader, which will be made available by the corresponding author P.C.P.N.

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