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The Formulation of Hypotheses in Mathematical Modelling Activities

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ABSTRACT

Background: The formulation of hypotheses in mathematical modeling activities, although it has been pointed out as one of the specificities of this kind of activities, it is still little discussed by the research in the area of mathematical modeling. **Purpose:** In this paper we look for influences on mathematical modeling activities arising from the formulation of hypotheses. **Design**: Our statements are based on a theoretical framework about Wittgenstein's philosophy and previous studies related to hypotheses' formulation as well as an empirical research. **Scenario and participants**: Modeling activities were performed by groups of students from different degrees. **Data collection**: In modeling classes, data were collected through audio and video recordings. The students also delivered the written records they produced. **Findings:** The findings indicate three categories for the formulation of hypotheses: hypotheses are formulated based on the students' view of the situation; hypotheses are based on students' experiences; hypotheses influence students' choices at different stages of the activity's development. **Conclusion:** The research concludes that the hypotheses determine the idealized situation, guide the mathematization and direct the students' actions in the development of the activity. In Wittgenstein's philosophical perspective, hypotheses are a way of perceiving reality and new experiences can lead to other formulations.

Keywords: Mathematical Modelling; Hypotheses; Wittgenstein's Philosophy.

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A Formulação de Hipóteses em Atividades de Modelagem Matemática

RESUMO

Contexto: A formulação de hipóteses em atividades de modelagem matemática, embora tenha sido apontada como uma das especificidades de atividades dessa natureza, ainda é pouco discutida nas pesquisas da área. **Objetivo**: Neste artigo investigamos desdobramentos para atividades de modelagem matemática decorrentes da formulação de hipóteses. **Design:** Nossas argumentações se fundamentam em um quadro teórico que considera elementos da filosofia de Wittgenstein e estudos anteriores relativos à formulação de hipóteses, bem como uma pesquisa empírica. **Cenário e participantes**: Atividades de modelagem são desenvolvidas por grupos de alunos de cursos distintos. **Coleta de dados**: Nas aulas com modelagem os dados foram coletados por meio de gravações em áudio e vídeo. Os alunos também entregaram seus registros escritos produzidos**. Resultados**: O processo analítico conduz a três categorias para a formulação de hipóteses: hipóteses são formuladas a partir de um modo de ver dos alunos com relação à situação investigada; hipóteses são fundamentadas em experiências dos alunos; hipóteses determinam as escolhas dos alunos nas diferentes fases do desenvolvimento da atividade. **Conclusão**: A pesquisa conclui que as hipóteses determinam a situação idealizada, orientam a matematização e direcionam as ações dos alunos no desenvolvimento da atividade. Na perspectiva filosófica de Wittgenstein as hipóteses são uma maneira de perceber a realidade e novas experiências podem levar a outras formulações.

Palavras-chave: Modelagem Matemática; Hipóteses; Filosofia de Wittgenstein.

INTRODUCTION

The identification of specificities of mathematical modelling activities has deserved attention from educators and researchers for some time. What some authors, such as Bean (2001), Djepaxhija *et al*. (2015), and Almeida (2014) have been highlighting is that there are indications that the formulation of hypotheses is one of these specificities.

Different fields of science recognise that the formulation, proof, and refutation of hypotheses cannot be dissociated from scientific activity in general. Regarding mathematical modelling, particularly, discussions on the subject have deserved some attention in investigations (Bean, 2012; Bassanezi, 2002; Almeida, 2014; Grigoraş, 2012; Djepaxhija *et al*., 2015; Seino, 2005; Galbraith & Stilmann, 2001; Chang *et al*., 2018). Different arguments can be perceived in these discussions, being on the agenda from the meaning of the word hypothesis to its inclusion in the development of mathematical modelling activities.

What the research indicates is that the formulation of hypotheses is a possibility in mathematical modelling activities to deal with information, specificities or characteristics of situations of the reality not yet known when the mathematical approach of this situation begins. Dealing with these unknown aspects and finding a way to overcome them can be addressed by formulating hypotheses (Maaβ, 2010).

As Galbraith and Stillman (2001) ponder, the hypotheses formulated cannot oversimplify the situation and should, above all, collaborate for interaction between the situation and the mathematics envisioned in mathematical modelling activities. However, formulating hypotheses may not be an easy task and, as Chang *et al*. (2018) point out, it is recurrently referred to as one of the greatest difficulties of students when they engage in modelling activities.

In this article, our interest is directed to the question: what are the developments for the mathematical modelling activity from the formulation of hypotheses? Our discussion, on the one hand, is based on elements of Ludwig Wittgenstein's philosophy of language and how it has been interpreted in the scope of mathematics education. On the other hand, we based ourselves on an empirical study in which mathematical modelling activities were developed by two groups of students, one from a mathematics teaching degree course and the other from a postgraduate course in mathematics education.

MATHEMATICAL MODELLING IN MATHEMATICS EDUCATION

In general terms, mathematical modelling refers to the investigation of a situation of the reality through mathematics(Almeida, 2018; Blum & Niss, 1991; Pollak, 2015; Meyer *et al*., 2011). [can we say here real situation?] The path of the modellers in this research can be guided by different configurations, considering the purposes of those who develop the activity and the perspectives of the mathematical modelling of these situations (Kaiser & Sriraman, 2006).

The development of mathematical modelling activities in the classroom at different levels of education presupposes that modellers (students and teachers) engage in activities in which they need to identify a problem situation of reality and formulate a problem associated with this situation, "deciding what to maintain and what to ignore in creating a mathematical model to deliberate on this problem and then deciding whether the results make sense in the face of the original situation" (Pollak, 2015, p. 267).

To the procedures identified by Pollak (2015), we usually associate what Almeida, Silva, and Vertuan (2012) call phases of mathematical modelling and consist of: integration (simplification, idealisation, data collection, and formulation of a problem); mathematisation (transition from natural language to mathematical language, formulation of hypotheses, and definition of variables); resolution (obtaining a mathematical model, use of concepts, theorems, procedures, and mathematical techniques); interpretation of results and validation (analysis of the model and confrontation of results in the face of the situation of reality). [inteiração = interaction, integration, complementation]

During the actions in these different phases, students or teachers share interests, problems, discussions, and make transitions between different languages. In this sense, Stillman *et al*. (2015) suggest that students and teachers, in fact, carry out the investigation of what the authors call an idealised situation and that results from reading and interpreting the real situation.

Therefore, it would be worth discussing how this idealisation occurs in modelling activities, which modellers do aiming at a mathematical reading and interpretation of non-mathematical situations. In this context, our look into the article is addressed to formulating hypotheses.

THE HYPOTHESES IN MATHEMATICAL MODELLING ACTIVITIES

The conception of mathematical modelling as an investigation of a situation of reality through mathematics that guides our reflections on the formulation of hypotheses in modelling activities comprises the idea already conveyed by Bean (2001) that a specificity of mathematical modelling activities is "the requirement of hypotheses and simplifying approaches as requirements in the creation of the model" (p. 53).

In the same direction, there are arguments by Stillman *et al.* (2015) considering that the situations of reality submitted to mathematical modelling in the classroom are situations idealised by modelling students and teachers. Massβ (2010) states that modelling activities in class usually begin with little information to support the mathematical modelling of the situation of the reality.

Thus, the discussion about the role of hypotheses in mathematical modelling activities, on the one hand, suggests the need to formulate hypotheses and, on the other hand, can turn to the developments of this formulation for activities of this type.

Starting from the idea that "modelling consists of the art of transforming problems of reality into mathematical problems and solving them, interpreting their solutions in the language of the real world," Bassanezi (2002, p. 16) states that "the hypotheses guide investigation" (p. 28). For the author, this formulation is anchored in assumptions such as the observation of data or information about the phenomenon, the comparison with the resolution of analogous problems, or even the experience of the modeller.

Referring to the introduction of mathematical modelling in mathematics classes, Almeida and Vertuan (2011) also point out the need for hypotheses, and consider that:

> [...] modelling has as its main contribution investigations carried out in the classroom that have the problem as a starting point, the intentionality in the search, the hypotheses as factors that stand in the way to indicate directions and in which different mathematical resolutions are undertaken with a view to solving a problem (Almeida & Vertuan, 2011, p. 22).

Grigoraş (2012) argues that formulating hypotheses is a component of the development of mathematical modelling activities that, although occurring in the initial stage of this development, can extend throughout the activity according to the complexity of the strategies, the methods used by the students, and the characteristics of the phenomenon under study.

In Seino's (2005) study, students from basic education become aware of the importance of the hypotheses they formulate when they develop a mathematical modelling activity. What the author points out is that this formulation can occur at different times during the development of the activity, however, the teacher's intervention is necessary for students to perceive the relationship between the hypotheses and the response found.

The interpretation that the formulation of hypotheses is a practice that allows students to overcome cognitive blocks that can prevent them from successfully carrying out the development of modelling activities is presented in Djepaxhija *et al.* (2015). The authors conclude that the definition of mathematical tools for the construction of the mathematical model of the situation can be attributed to the hypotheses.

As Galbraith and Stillman (2001) discuss, the hypotheses act as guesses that should be part of what the authors call genuine modelling activities. Although this formulation often occurs in a specific phase of modelling, from the perspective of these authors, its essential function lies in the integration of mathematics with the situation of reality as a key point for progress in the development of the mathematical modelling activity.

In this article, we propose a reflection on the formulation of hypotheses and even on the word *hypothesis* itself in the context of the development of modelling activities considering elements of the philosopher Ludwig Wittgenstein's itinerary of thought on language, more specifically on what he characterises as *language games*. For Wittgenstein, the different practices in which language is used, or contexts in which it is included, are called language games, so that he states “I will also call ‘language games’ the set that comprises the language and the activities with which it is intertwined” (Wittgenstein, 2013, § 07). Elsewhere the author points out “the meaning of a word is its use in the language” (Wittgenstein, 2013, § 43) and he called those uses language games. The actions with language, in this sense, are configured as language games, such as calculating, enunciating a poem, singing a song, among others. Mathematics and mathematical activity, especifically mathematical modelling activities, are made up of language games that modellers need to deal with.

Regarding the word hypothesis, different meanings seem to be configured as a result of the language games in which the term appears, considering what dictionaries present to refer to the hypothesis or speak of its meaning. According to Abbagnano (2007), the hypothesis refers to an utterance that can only be proven, examined, verified indirectly through its consequences. For Japiassú and Marcondes (2008), a hypothesis is a provisional explanation of a phenomenon that must be proven by experimentation. In Houaiss (2009), a hypothesis can be a proposition that admits a principle from which a set of consequences can be deduced.

Taking into account these different meanings, we could ask: when the context is mathematical modelling, how to talk about the meaning of the hypotheses? Are the hypotheses used in the development of mathematical modelling verifiable through its consequences?

Some reflections on these issues may come from Wittgenstein's thinking regarding the hypotheses. When dealing with the nature of the hypotheses in his Philosophical Grammar, the philosopher states that a hypothesis can result from our experiences and, precisely for this reason, it can be modified, it can be replaced, and exemplifies his conjecture as follows:

> If our experiences result in points on a straight line, the proposition that those experiences are multiple views of a straight line is a hypothesis. Hypothesis is a way of perceiving this reality. A new experience may coincide with it or not, and possibly make it necessary to modify the hypothesis (Wittgenstein, 2003, p. 169).

In this context, formulating a hypothesis in mathematical modelling activities could also be associated with the modellers' experiences, either with the situation of reality, or with mathematics, or with mathematical modelling practices. The experience would result in ways of seeing to foster formulating hypotheses. Thus, *a priori*, there would be no right or wrong hypotheses. However, the modeller' experience and his/her information about the situation may lead to formulations whose adequacy and veracity can be confirmed when the model is considered adequate or when it meets the interests of the modeller and a community.

The modellers' experiences are also mentioned in Chang *et al*. (2018), who suggest that the hypotheses formulated can be separated into two groups: non-numeric, which relate to the conditions of the situation investigated and involve extra mathematical knowledge; numerical, which aim to overcome the lack of quantitative information about the situation.

Considering it necessary to confirm the veracity of a hypothesis so that it is useful in the language game in which we are, Wittgenstein says:

> I may wonder whether the body I see a sphere, but I cannot wonder that, from here, it seems to be something like a sphere. The mechanism of the hypothesis would not work if the appearance were also doubtful so that we could not verify beyond doubt even a facet of the hypothesis. If there was a doubt in the case, what would eliminate the doubt? If this connection were also loose, there would be nothing with which to confirm a hypothesis and it would hover in the air entirely, entirely aimless (and, therefore, useless) (Wittgenstein, 2003, p. 171). [não tenho como verificar se há uma versão em inglês, fiz a minha própria tradução]

An interpretation of Wittgenstein's assertion leads us to consider that, even if hypotheses are formulated in a state of doubt of the modellers, they need to be permeated with some certainty, because if doubt prevails, actions are not defined. It seems to be with this understanding that in Almeida (2014) hypotheses in mathematical modelling activities are characterised as well-founded assumptions, the foundation of which derives from indications, facts, and information related to the situation of the reality that is under investigation. We can also consider that the idea of guesses suggested by Galbrath and Stillman (2001) is supported by Wittgenstein's reasoning, considering that, according to them, on the one hand, guesses include modellers' experiences and on the other, simplify the problem so that, although doubts remain, there is some certainty to guide the mathematical modelling of the situation.

The migration of Wittgenstein's ideas to the scope of mathematics education leads us to consider that, on the one hand, the formulation of hypotheses in mathematical modelling activities cannot occur unrelated to the characteristics of the situation. On the other hand, it also requires some experience or some anticipation, as considered by Niss (2010) and Almeida (2018), related to both mathematical knowledge, information about reality, and their experience with the development of mathematical modelling activities. These aspects favour identification in relation to the functioning of the mechanism of the hypothesis, as Wittgenstein considers.

Taking into account these approximations glimpsed between Wittgenstein's thought and hypotheses, the empirical research aims to foster and expand the perspectives pointed out about formulating hypotheses in the development of mathematical modelling activities and give indications of the developments for the activity resulting from this formulation.

METHODOLOGY[1]

The empirical research carried out includes two mathematical modelling activities. One of them was developed by a group of students in the Mathematical Modelling module offered by a postgraduate programme in mathematics education at a public university. In this module, attended by 20 students, the different groups developed activities, and the theme of interest was defined by each group. In this article, we refer to the activity of one of these groups and the choice of this group stems from the quality of the data obtained and from the emphasis of this group on formulating hypotheses. The other

[1] The data used in the research are not part of a project submitted to the Ethics Committee. The authors are responsible for submitting data, and the journal Acta Scientiae is exempt from any responsibility. According to Resolution No. 510, of April 7, 2016, of the National Health Council of Brazil, full assistance and eventual compensation for any damage resulting from any of the research participants is an authors' attribution.

activity - the theme of which was suggested by the teacher - was developed by 21 students of a mathematics teaching degree course in the module Mathematical Modelling from the Mathematics Education Perspective. In this module, the same activity was developed by all groups of students. In the article, we included the development of one of these groups. The group was chosen due to the quality of the data obtained and the information of the group about the issue investigated in this article. Both modules were taught by one of the authors of this article.

From a methodological point of view, this is a qualitative research. The analytical process is related to the data that were obtained from the reports of the activities delivered by the students and transcripts of audio and video recordings made during the classes in which the activities were developed, and the presentation of the works of each of the two groups to all students in each of the disciplines. The interpretative analysis of those data in the light of the referred theoretical framework leads us to characterise categories related to the developments for the activities resulting from students' hypotheses formulation.

The first activity

This activity refers to the monitoring of the heart rate of individuals during physical exercises. It was developed by a group of students in the Mathematical Modelling module offered by a postgraduate programme in mathematics education. According to their report, the students felt motivated to know how each person's physical fitness influences performance in physical exercises. Thus, from information on specialised *websites* and books, they realised that there are different types of tests that can be used in this investigation. They opted for one of these tests, known as the Léger and Lambert test[2] and set as the objective of the activity to build a mathematical model that describes the variation of an individual's heart rate during the Léger and Lambert test, popularly known as the 20-m shuttle run test.

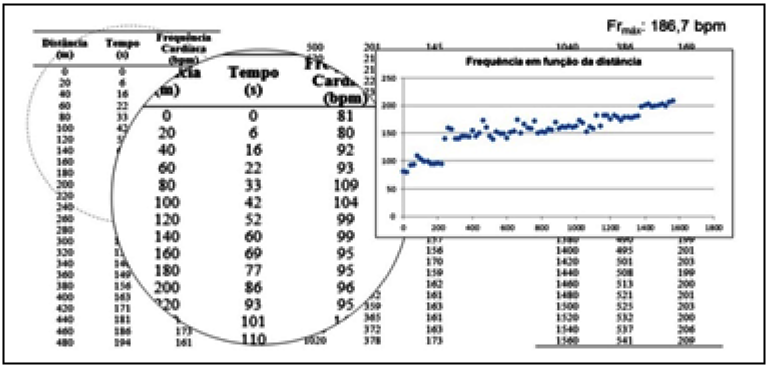
The group faced the need to collect data from people who had different habits regarding physical activities. Therefore the students themselves constructed the data through an experimental procedure. Among students involved and their families, they chose four subjects (runners), two sedentary and two who declared to exercise regularly, one of them being a professional in physical education area, who knew the Léger and Lambert test. With materials

[2] Details of the method are in Duarte & Duarte (2001).

such as stopwatch, frequency meter, tape measure, adhesive tape and audios, the group of students demarcated the distance on a sports court as indicated by the instructions for the test and recorded an audio and video for each individual who took it. For this article, we bring the data students' collected from a runner (Figure 1).

Figure 1

Data collected from a runner's performance in the Léger and Lambert Test



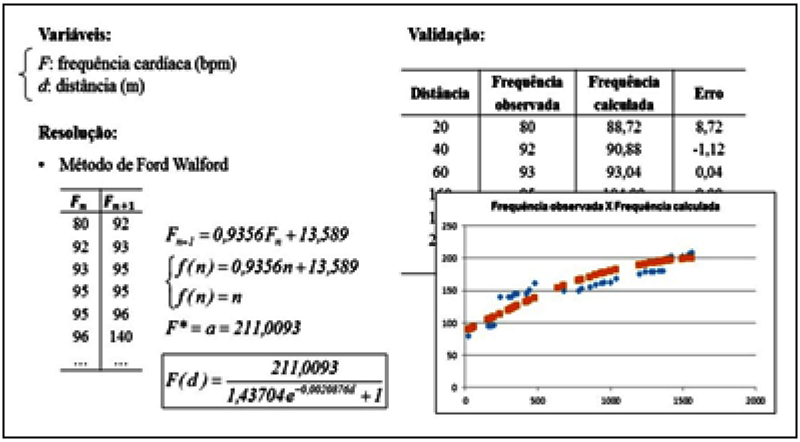
The students' mathematical approach of this situation began with the interpretation of the data obtained, providing information about the situation. Their report suggests that this first interpretation of the data resulted from the formulation of three hypotheses: $_{H1.1}$: There is a regularity in the variation of heart rate; $_{H2.1}$: There is a relationship between the variation of heart rate and the distance the runner traveled; $_{H3.1}$: The runner's heart rate increases and tends to stabilise as it approaches his/her maximum.

Using the data in Figure 1 and these hypotheses, the students mathematised the situation and built a mathematical model using the Ford-Walford method as[3] shown in Figure 2. The validation of the model in this case was done by the students by comparing the observed (blue dots) with data obtained by the model (yellow curve), as shown in the image in Figure 2. [the observed...what?]

[3]Method to determine the stability value in asymptotic models and described in detail in Bassanezi (2002).

Figure 2

Mathematisation and constructed mathematical model



An interpretative analysis of the consequences of those hypotheses for the development of the activity leads us to some inferences between the relationship of the hypotheses with the other students' referrals.

Hypothesis $H_{1.1}$ (There is a regularity in the variation of the heart rate) seems to be based on the characteristic of the heart rate phenomenon: the need for a regularity of the beats. In fact, although the students can observe some variability in heartbeat, in this hypothesis, they suppose the movement should be regular, considering the runner's physical preparation. We can say that $H_{1.1}$ aims to elucidate characteristics of the problem to be investigated, thus being a way of seeing the situation or perhaps an interpretation. And, according to Wittgenstein (2013, p. 276), “to interpret is to think, to act; seeing is a state”. The action, in this case, makes it possible to think about which mathematics to use to solve the problem and how to use it to understand the variation of the heart rate in the Léger and Lambert test. In this context, the first hypothesis was formulated as a baseline for reasoning, a path to follow to solve a problem, i.e., a guide for research as argued in Almeida (2014).

Regarding hypothesis $H_{2.1}$ (There is a relationship between the variation of heart rate and the distance traveled by the runner), it clearly contributes to the mathematical referral of the situation. Indeed, as shown in Figure 1, time, distance traveled, and heartbeat are the three variables included in the collected data in this case. It is possible to affirm that this hypothesis guides the students' noticing to define which mathematics they will use in the elaboration of the mathematical model, since unsing the three variables would imply constructing a model based on a function of two independent variables, or, in two functions of an independent variable (variation of heart rate as a function of the distance traveled; distance traveled as a function of the time spent).

The students inform both in the report and in the presentation of the activity to their colleagues that the Léger and Lambert test have standardised stages, thus, they consider it appropriate to elaborate a model considering the variation of the heart rate as a function of the distance traveled. This fact meets Almeida and Vertuan's (2011) indications that the hypotheses are factors that stand in the way to indicate directions and in which different mathematical resolutions are undertaken to solve the problem. Therefore, we infer that $H_{2.1}$ assumes in the activity the role of delimiting and simplifying the mathematical situation to be solved by the students. What mathematics should be considered stems from the modellers' ways of seeing the situation, their observation and experience, and the information they have about the situation.

From the two initial hypotheses, students enter the construction of a mathematical model guided by the third hypothesis formulated, $H_{3.1}$ (The behavior of the heart rate is increasing and tends to stabilize as it approaches the runner's maximum heart rate). In presenting the activity to all students in the module, the students in the group justify this hypothesis: “We had to consider that the beat does not increase indefinitely and this in mathematics means thinking of a function that has a [4] horizontal asymptote” (audio transcription of the activity presentation). Another student then adds: “That is why we chose a function of this type,”, pointing to the *slide* with the information: $F(d)=\frac{a}{b.e^{-\lambda d+1}}$, in which $a,b,\lambda \in R$ e F is the heart rate (bpm), d is the distance (m). (audio transcription of the activity presentation) Using the data in Figure 1, the students built the mathematical model for the situation as shown in Figure 2.

Thus, we can consider that hypothesis $H_{3.1}$ refers to the use of mathematical language available to modellers, a conventional language of an objective nature, with specific rules and grammar. This hypothesis is seen as a law that guides the students. We can say that this law reflects an expectation of

[4]The horizontal asymptote to which the student refers in this activity is related to the value of stability for the heartbeat.

the group regarding which mathematics to use to build the mathematical model and, therefore, an expectation to obtain a solution to the problem.

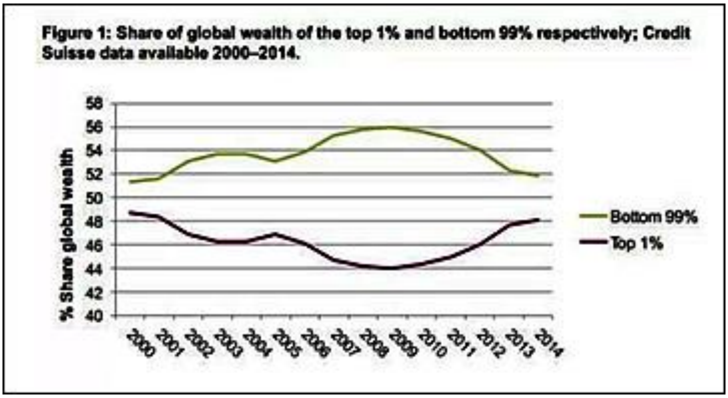
The image in Figure 2 indicates that the behaviour of the constructed mathematical model (in yellow) does not correspond, point by point, to the data collected during the corridor test (blue dots), which corroborates the idea that a hypothesis in this case leads to an approximation to notice the situation and that can be modified according to the experience of the modeller, with the way of seeing and interpreting this situation. They observed that the limiting value (the asymptote) for the subject's heart rate obtained through a mathematical method - 211 bpm - is higher than the maximum estimated heart rate according to the health area literature, which indicates the value of 186.7 bpm.

The second activity

This activity was developed by students of the mathematics teaching degree course during four classes of the module Mathematical Modelling from the Mathematics Education Perspective in the second semester of 2019. Students developed the activity in groups. We are referring here to the development of one of these groups.

Figure 3

Image taken to the students by the teacher (Exame magazine of January 19, 2015)



The activity originated in a situation presented to the students by the teacher as shown in Figure 3. This image, published in Exame magazine, follows a report on the inequality in the distribution of wealth in the world.

According to the report, the graphic intends to draw attention to the fact that inequality has grown from 2000 to 2014. According to the report, the image should highlight the impact of the distribution of global wealth, indicating a relationship between the wealth of the 1% of the richest people in the world, in relation to the wealth of the other 99% of the population.

From interpreting the situation presented in this image, a mathematical work was triggered, aiming to analyse the behaviour of the distribution of wealth. In other words, as in the previous activity, in this activity, there is a clearly defined problem with respect to the phenomenon to be investigated. Instead, what students would do is present, through mathematics, an analysis of the situation seen in the image.

The students involved in this research envisioned a dynamic for the image by formulating four hypotheses throughout the development of the activity and making use of GeoGebra *software* resources for the mathematical approach of the situation.

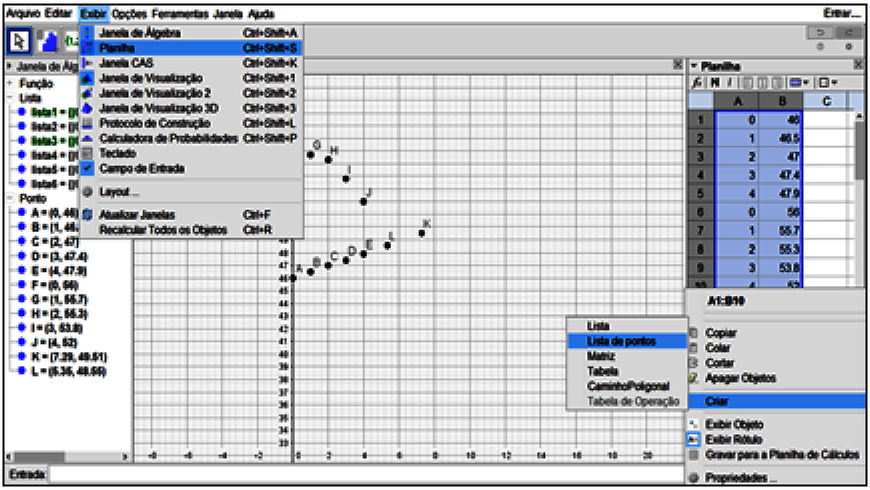
Initially, the students defined the hypothesis $H_{1,2}$: there is a possibility that the wealth of the 1% richest will reach the wealth of the other 99% of the population, which, according to the report delivered by the students, is "due to the feeling that the image for the period from 2000 to 2014 indicates that there is a possibility that these two curves will intercept some year after 2014 if there is no change in the behaviour of the situation" (students' report).

This hypothesis gives rise to the phase of mathematisation, when they must see beyond the image per se and notice which mathematics can be used to understand how this situation of the distribution of wealth will behave over time. Therefore, the hypothesis acts as a guide for the application of mathematical rules from the data on the situation identified in Figure 3.

To guide the construction of a mathematical model capable of enabling the estimates for the approximation of the riches of the two groups of the population, the students defined a second hypothesis: $H_{2.2}$: From the information in the image of Figure 3 we will adjust linear models to the data. The construction of the model was mediated by the resources of the GeoGebra *software* and began with the students' insertion of data on the situation in the *software,* as shown in Figure 4. In the image, the values in column A (0,1,2,3,4) represent, respectively, the years 2009 to 2014, and the values in column B an approximation according to the graph of the wealth of the richest 1% and the other 99% of the population.

Figure 4

Mathematisation of the situation using GeoGebra



Using the Curve Fitting Toolbox in the *software* , the students built the linear models. [verificar] For the evolution of the riches of the 99% of the population, they obtained $f(x)= -1.05x +43.92$, and for the evolution of the richest 1% of the population, the model obtained was $g(x)= -0.97x +56.34$ (Figure 5 (a)). These models led students to conclude that: "In 2016, the wealth of the two groups already matched, i.e., from that year, the wealth of the 1% richest exceeded that of the other 99% of the world's population" (students' report).

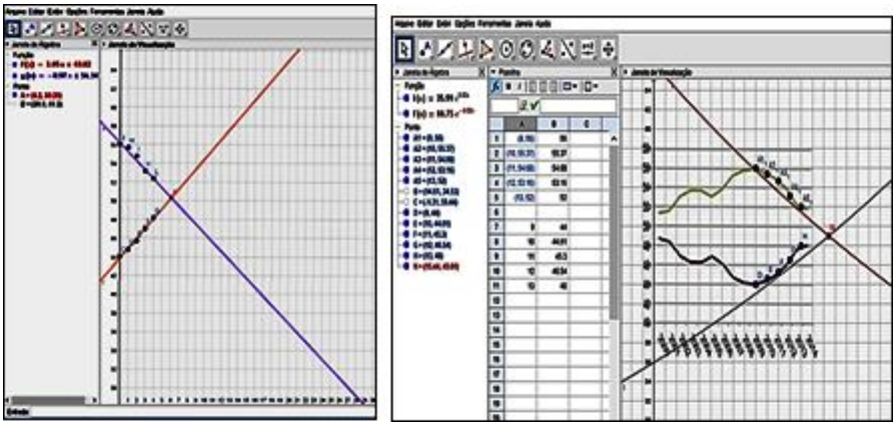
As stated during the presentation of the work in class (presentation recorded in audio and video), "taking advantage of the resources of the *software*, we can do other simulations and therefore we define another hypothesis" (transcription of one student's audio recording). The hypothesis to which the student refers is: $H_{3,2}$: Let us suppose an exponential behaviour for the distributions of wealth over time, from the year 2009.

Using the same tool in GeoGebra, they found models $h(x)=67.75e^{-0.02x}$ and $j(x)=67.75e^{-0.02x}$ to describe the evolution of the wealth of the richest 1% and the evolution of the wealth of the other 99% from 2009, respectively (Figure 5 (b)). Also in this case, equaling the two models, the students conclude that, in 2006, the wealth of the two groups was equal already.

Based on the models, the students infer: "We can conclude that the distribution of wealth tends to become increasingly unequal over time" (students' report). To investigate how this situation could behave over time, the students defined more hypotheses: $H_{4,2}$: The behaviour of the distribution of wealth will not change sharply over the next 10 years. This hypothesis aims to support the predictive purpose of the mathematical models obtained. To put it another way, predicting how the distribution of wealth will behave in the coming years is a consequence of this hypothesis.

Figure 5

(L) Linear models. (R) Exponential models



In this activity, the hypotheses defined by the students guided the use of mathematical rules for the interpretation of the situation. From them, follows what Moreno (2003) identifies as *linguistic appropriation* , which we address here in relation to the linear and exponential models built, the use of the *software* , and the significance resulting from the use of the models for the estimates and analysis of the situation of the wealth distribution.

DISCUSSION AND RESULTS

In this article, our reflections on the developments resulting from the formulation of hypotheses for mathematical modelling activities take into account the activities developed by two groups of students and reveal an interpretation of elements of Wittgenstein's philosophy of language. In this context, mathematical modelling activities involve linguistic actions and articulations related to dialogues between mathematics and the situation of reality.

We can infer from the analytical effort undertaken that the hypotheses the students formulated, while having different origins, also have different purposes for the development of the activity. On the one hand, the study corroborates indications from the literature such as Bassanezi's (2002), which indicate that the hypotheses direct the investigation, and results by Almeida and Vertuan (2011), which indicate the need for the hypotheses as "factors that stand in the way to indicate directions and in which different mathematical resolutions are undertaken" (p. 22). Those indications could suggest that the formulation of hypotheses is established in the initial stages of the activity.

On the other hand, however, the analysis undertaken in this article indicates that from the formulation of hypotheses the other actions of the students can be configured, expressing different ways of seeing the situation and a possible solution. These ways of seeing are not private interpretations, but direct students to publicise the use and application of mathematical rules for specific situations within each activity. In this context, the hypotheses guide students to express the mathematical techniques learned, as well as the rules of the situation they learned through the mathematical modelling of the situation. They differ in the nature of each context, the 'situation investigated' and the 'use of mathematics,' as well as the linguistic relationships provided by the hypotheses within the mathematical modelling activities.

In the first activity of mathematical modelling, the students' three hypotheses have specificities, either in their origin or the unfoldings for the activity to be carried out. Hypothesis 1, based on the students' knowledge about the situation, acts as a guide for understanding the problem. In fact, admitting that there is a regularity in the variation of the heart rate ($H_{1,1}$) would be the first step to investigate how the variation of this frequency behaves. The formulation of $H_{1,2}$ (There is a relationship between the variation of heart rate and the distance travelled by the runner) is based on what the students studied about the phenomenon, knowing that the frequency can be described directly considering only the distance travelled (time being an implicit variable in this case). Its consequence for the development of the activity would, however, be decisive. Not explicitly considering the time variable would bring a specific mathematical description, different from the one if time were also a variable considered in the situation.

The third hypothesis, in turn, corresponds to a reading of the collected data. It indicates the referral for the construction of the mathematical model.

Admitting an asymptotic behaviour of the data defined the choice of the mathematical model to understand the variation of the heartbeat. The mathematical model in that case is a symbolic construction that reflects the relationships and characteristics considered relevant to the situation.

The use of mathematical rules to construct models in this activity is based on mathematical conventions and, in this case, on a language game whose certainty occurs within a community. The use of mathematical rules is put by Wittgenstein (1996; 2013) as the use of grammatical propositions that express a conventional certainty. Thus, formulating hypotheses, initially based on empirical data, leads to grammatical research whose rules to be followed are mathematical certainties. The possible unfolding from the hypothesis formulation to the deduction of a mathematical model evidences an important characteristic of mathematical activity: the use of a normative, regulated language and, in Wittgensteinian words, 'grammatical,' which differs, for example, from the nature of the use of hypotheses in the natural sciences.

The hypotheses formulated by the students are associated with the reading of the data collected and express a way of seeing the situation from their previous experiences, either with the phenomenon investigated or with the investigation of regularities from the data tabulation. In a classical approach to Wittgensteinian philosophy, ways of seeing and interpreting are subjects of reflection. Wittgenstein suggests that the way of seeing a figure, a triangle, for example, is influenced by habit and education (Wittgenstein, 2013, p. 263). Thus, in the academic context in which these students are, noticing the situation is loaded with experiences and ways of seeing that stem from this context. The formulation of the hypotheses in this activity is loaded with such students' experiences.

The uses of language are a result of the linguistic practices experienced by the students in the context of a postgraduate course. The fact that a hypothesis indicates to students one direction or the other may be associated with what Wittgenstein describes as a habit of following a rule or applying a rule, which is not based on an interpretation, as it is not something private, but based on the training and instruction previously received (Wittgenstein, 2013, § 201-203). In this sense, formulating hypotheses puts students in practice with mathematical rules previously seized, showing a path to applying the rule. The resulting mathematical model, however, has some degree of generality, i.e., it can be useful in situations beyond that for which it was built, thus fostering the students' experience.

In activity 2, which consists of the analysis of a phenomenon that students face through an image and magazine report, the first hypothesis $H_{2.1}$ (There is a possibility that the wealth of the richest 1% will reach the wealth of the other 99% of the population) formulated by the students is an interpretation of the data associated with the situation and, therefore, a way of seeing the situation of the wealth distribution. This interpretation determines how mathematics would be used to analyse the situation, inciting the referrals that should be established in the mathematisation of the situation.

Once the students decided that a computational resource (*GeoGebrasoftware*) would be the tool to mediate the mathematisation of the situation, the hypotheses $H_{2.2}$ (From the information in the image of figure 3 we will adjust to the data linear models) and $H_{3.2}$ (Let us suppose an exponential behaviour for wealth distributions over time, from the year 2009) are formulated to indicate clearly which mathematics they would use to ponder the wealth distribution. In this sense, these two hypotheses take into account the behaviour of the information revealed by the image in Figure 3. These hypotheses, in turn, have a different purpose from the previous one and aim to direct the formulation of the model and the mathematical processes that should guide the mathematical modelling of the situation. They are structured from previous experiences either with mathematics or with the *software* used. Thus, although it is not possible to characterise the hypotheses as what Wittgenstein calls fixed certainties, they are also not formulated loosely and can be considered as well-founded assumptions, as suggested by Almeida (2014).

Despite the students' mathematisation regarding the lack of equity in the distribution of wealth, the social and critical nature of this situation led the students to go beyond the first interpretation and deliberate on the predictive possibility of the mathematical models built. This use of the model for forecasts was guided by a new hypothesis, $H_{4.2}$ (The behaviour of the distribution of wealth will not change sharply over the next 10 years). This hypothesis denotes the students' interest in expanding the discussion about the situation made possible until then. So, we can infer that this hypothesis is associated with students' expectations regarding the function of the mathematical model in this activity, as suggests the transcription of part of a dialogue between students and teacher during the class in which the activity was carried out (A_1 and A_2 are two students in the group and P is the teacher):

> P: But did you define this hypothesis after building the two mathematical models?

A_1: Yes, teacher, I mean, we wanted to know how this situation would be in the future.

A_2: As we had already seen the riches of each group of the population, we wanted to predict how this will go on...

P: Got it.

An analysis of the students' formulation of hypotheses in the two activities allows us to consider that, while students used their experiences with the situation and with mathematics to formulate the hypotheses, they also worked as the guideline for reading or investigating this situation. Our analysis leads us to conclude that the developments for the modelling activity result from the specificities of the situation recognised by the students from their personal experiences and from their interest in presenting results -through the interlocution between situation and mathematics- for what they proposed to investigate in mathematical modelling. The analysis of the unfoldings that formulating hypotheses provided in the two activities leads us to consider that three categories can be characterised regarding the relevance of those formulations in the mathematical modelling activities, as follows.

(1) The hypotheses are formulated from the students' way of seeing with respect to the situation investigated in the mathematical modelling activity, and, therefore, lead to a dated and dialogical solution. Thus, the hypotheses determine the idealised situation that the students will investigate.

(2) The hypotheses are based on students' experiences. Based on this experience, students define hypotheses in relation to two aspects: (a) hypotheses that determine which mathematics will be useful or necessary and that can provide the interlocution between the situation of reality and mathematics through the mathematical model; (b) hypotheses that determine the procedures and tools internal to mathematics and that will subsidise the construction of the mathematical model and its validation regarding the situation of reality.

(3) The hypotheses determine the students' choices in the different phases of the activity's development. With this function, the hypotheses guide the students' actions. Thus, data can be complemented, impasses related to mathematical procedures or the specificities of the situation are overcome, and partial solutions are reviewed and complemented from defined hypotheses.

Those unfoldings are associated with the way of conducting mathematical modelling activities and the ways of operating with language within an activity in which the application of conventional rules within mathematics often occurs.

To the extent that hypotheses are formulated in mathematical modelling activities, they expose what students see of the situation of reality and, at the same time, encourage students to deepen their information status about the situation. In this sense, in a mathematical modelling activity, a hypothesis is not like an utterance that can only be proven, examined, verified indirectly through its consequences (Abbagnano, 2007). Nor is a modelling hypothesis a provisional explanation of a phenomenon that must be proven by experimentation (Japiassú & Marcondes, 2008). However, the hypotheses bring certainty in them, as Wittgenstein (2003) suggests, and this certainty is reflected in the results of the modelling developed.

Thus, the nature of a hypothesis in mathematical modelling has characteristics of a well-founded assumption, as Almeida (2014) calls it. Indeed, the data indicate that, to some extent, the hypotheses determine which mathematics will be used and how this use will be, as discussed in Almeida (2018). Above all, the modeller needs to provide information about the situation, about mathematics, to ensure that the mechanism of the hypothesis works, as Wittgenstein (2003) ponders, and the mathematical rules are used in line with the language game that is configured in the modelling activity. So, following the rules Wittgenstein (2013) refers to does not disconnect from mastering techniques and the use of a reference framework such as mathematics, to act linguistically by investigating a situation of reality.

The validation of the results obtained by the students can also guide the functioning of the hypotheses, as it works as a criterion for correcting the techniques used within the mathematics language game. This seems to have been, for example, the referral of students to study the behaviour of heartbeats in runners submitted to the Léger and Lambert test.

In the second activity, from formulating hypotheses and constructing models and their use to analyse the situation, the students could experience what Niss (2015) calls meta-validation associated with the prescriptive role of mathematical models. In the mathematical modelling in which the models propose to support predictions with respect to the investigated situation, we want "to identify possibilities of transforming the world rather than just understanding it" (Niss, 2015, p. 69). In the case of the activity of distribution of wealth, this meta-validation analysing the consequences of the results obtained by the model made students formulate a new hypothesis ($H_{4.2}$ The behaviour of the distribution of wealth will not change sharply over the next 10 years) and expand their discussion on the social and political issues immersed in this situation.

The students' meta-validation does not disqualify previous mathematical rules used them. Within mathematical modelling activities, the application of the rules occurs in a specific context of use and, in this case, the hypotheses play a guiding role for how the mathematical rule will be applied in the context. New hypotheses can lead to the use of other rules or even another use of the same rule.

In other words, about the mathematisation of situations, the formulation of the hypotheses delimits an idealised situation, as characterised by Stillman *et al*. (2015) and Djepaxhija *et al*. (2015), based on the real situation, so that the hypotheses establish conditions so that a mathematical guise can be given to the situation. Thus, the hypotheses are a public interpretation of the use of mathematical techniques that makes sense in a given context, in line with Wittgenstein (2013).

The hypotheses, in addition to a contextual demand considering the situation of reality and mathematics, as Galbraith and Stillmann (2001) suggest, also required from the students means to elucidate the interlocution they intended, using mathematical rules to interpret reality through mathematical models. However, the mathematical models thus constructed were agreed within the students' way of life so that one cannot decide on true mathematical models or false mathematical models, as Bassanezi (2002) suggests.

This leads us to affirm that the mathematical modelling activities have a referral that agrees with the hypotheses on which they are based. New experiences of modellers can lead to new hypotheses, to new ways of seeing and understanding, through mathematics, a situation of reality.

Thus, as agreed in science in general, also in the language game of mathematical modelling it is illogical to speak of true or false hypotheses, but we can speak of hypotheses as factors that indicate directions and guide the modeller. In this language game, the hypotheses indicate ways of seeing, interpreting, or pointing out intervention strategies on situations. Thus, “Hypothesis is a way of perceiving reality, because a new experience may coincide with it or not...” (Wittgenstein, 2003, p. 169).

FINAL CONSIDERATIONS

The investigation of the unfoldings for mathematical modelling resulting from the formulation of hypotheses was mediated by an empirical study in which mathematical modelling activities were developed by two groups of students from two different academic modules. The analytical process of these activities is based -in addition to research in mathematics education that deals with the theme- on elements of Wittgenstein's philosophy of language and its repercussions on mathematics education.

The analysis undertaken related to the mathematical modelling activities developed by the two groups allows us to consider that students' hypotheses formulation signals a way of seeing, is anchored in their experiences, and provides elements for subsequent actions. Thus, changes in the hypotheses transform the actions and the resulting conceptual construct, and the understanding of the situation under study.

The Wittgensteinian perspective used in the article leads us to point out that the hypothesis is a way of perceiving reality and does not stabilise in a complete state of doubt. Thus, if in mathematical modelling activities the need to formulate hypotheses aims to supply some lack of information about the situation, it is also not free from the use of the information that the modeller has about the situation.

Identifying categories in relation to the unfoldings of formulating hypotheses for the development of modelling activities, while elucidating the role that this formulation has in the development of these activities, is a result not yet found in the literature and can, to some extent, guide teachers when including modelling activities in their classes, attending to the relevance of this formulation.

In the empirical research that supported this article, we considered different groups of students involved with mathematical modelling activities for different situations. In this sense, the students' experiences were also diverse. Future research can change this picture and obtain results on the influence of specific personal experiences on formulating hypotheses.

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AUTHORSHIP CONTRIBUTION STATEMENT

LMWA was the professor of the modules in which the mathematical modelling activities were developed. Collaboratively, the three authors appropriated the theoretical foundations of the article and carried out data collection and its analysis. The text was also drafted in a joint and collaborative manner.

DATA AVAILABILITY STATEMENT

The data supporting the results of this study will be made available by the corresponding author, LMMA, upon reasonable request.

REFERENCES

Abbagnano, N. (2007). *Dicionário de Filosofia*. Martins Fontes.

Almeida, L. M. W. (2014). The 'Practice' of Mathematical Modeling Under a Wittgensteinian Perspective. *International Journal for Research in Mathematics Education*, *2*, 98-113.

Almeida, L. M. W. (2018). Considerations on the use of mathematics in modeling activities. *ZDM*, 50(1), 19-30.

Almeida, L. M. W. & Vertuan, R. E. (2011). Registros de representação semiótica em atividades de Modelagem matemática: uma categorização das práticas dos alunos. *Unión (San Cristobal de La Laguna*), *25*, 109-125.

Almeida, L. M. W., Silva. K. P., & Vertuan, R. E. (2012). *A modelagem matemática na educação básica*. Contexto.

Bassanezi, R. C. (2002). *Ensino-aprendizagem com Modelagem Matemática*. Contexto.

Bean, D. (2001). O que é modelagem matemática? *Educação Matemática em Revista*, *8*(9/10), 49-57.

Bean, D. (2012). As premissas e os pressupostos na construção conceitual de modelos. In: *Anais do Seminário Internacional de Pesquisa em Educação Matemática*, SBEM, (p. 1-22).

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects – state, trends and issues in mathematics instruction. *Educational Studies in Mathematics*, Dordrecht, *22*(1), p. 37-68.

Chang, Y. P., Krawitz, J., & Schukajlow, S. (2018). Comparing German and Taiwanese secondary school students' knowledge in solving mathematical modelling tasks requiring their assumptions. *ZDM*, *52*, 59–72.

Djepaxhija *et al*. (2015). Exploring grade 9 students'assumptions making when mathematizing. *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (CERME 9).* (p. 848-854).

Duarte, M. F. S., & Duarte, C. R. (2001). Validade do teste aeróbico de corrida de vai-e-vem de 20 Metros. *Revista. Brasileira de Ciências e Movimento*, *9*(3), 7-14.

Galbraith, P., & Stillman, G. (2001). Assumptions and Context: Pursuing their role in modelling activity. In J. Matos, W. Blum, K. Houston, & S. Carreira (Eds), *Proceedings of the Modelling and Mathematics Education ICTMA 9: Applications in science and technology*. (p. 300–310).

Grigoraş, R. (2012). Mathematising through hypotheses and assumptions. *Proceedings of the 12th International Congress on Mathematical Education,* COEX, Seoul, Korea, 1-10.

Houaiss, A. (2009). *Grande dicionário da língua portuguesa*. Objetiva.

Japiassú, H., & Marcondes, D. (2008). *Dicionário Básico de Filosofia*. 5.ed. Zahar.

Kaiser, G., & Sriraman, B. (2006). A global survey of international perspectives on modelling in mathematics education. *ZDM, 38*(3), 302-310.

Maaβ, K. (2010). Classification scheme for modelling tasks. *Journal für Mathematik-Didatik*, *31*(2), 285-311.

Meyer, J. F. C., Malheiros. A. P., & Caldeira, A. D. (2011). *Modelagem em educação matemática*. Autêntica.

Moreno, A. R. (2003) Descrição fenomenológica e descrição gramatical – ideias para uma pragmática filosófica. *Revista olhar*, *4*(7), 93-139.

Niss, M. (2010). Modelling a crucial aspect of students' mathematical modelling. In R. Lesh *et al*. (Eds.), *Proceedings of the Modelling Students' Mathematical Modelling Competencies (ICTMA 13)*. (p. 43–60). Springer.

Niss, M. (2015). Prescriptive Modelling – Challenges and Opportunities. In: G. A. Stillman, W. Blum, & M. S. Biembengut (Eds). *Mathematical Modelling in Education Research and Pratice*. (p. 161-171). Springer.

Pollak, H. O. (2015).The Place of Mathematical Modelling in the System of Mathematics Education: Perspective and Prospect. In: Stillman, G., Blum, W., & Biembengut, M. S. (Eds.) *Mathematical Modelling in Education Research and Practice*: cultural, social and cognitive influences. (p. 265-276), Springer.

Seino, T. (2005). Understanding the role of assumptions in mathematical modelling: Analysis of lessons with emphasis on the 'awareness of assumptions'. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Proceedings of the Building connections: Research, theory and practice* (MERGA 28), 2. (p. 664–671).

Stillman, G.A., Brown, J. P., & Geiger, V. (2015). Facilitating mathematisation in modelling by beginning modellers in secondary school. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds), *Mathematical Modelling in Education Research and Practice*: *Cultural, Social and Cognitive Influences*. (p. 93–104). Springer.

Wittgenstein, L. (1996). *Remarks on the foundations of mathematics*. The MIT Press.

Wittgenstein, L. (2003). *Gramática filosófica*. Loyola.

Wittgenstein, L. (2013). *Investigações Filosóficas*. 8. ed. Vozes.