Modelling Improper Integrals, a Case Study

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ABSTRACT

Background: University students show little clarity in applying content related to improper integrals due to a lack of meaning that does not allow them to connect them with the environment and everyday life problems. **Objective:** To recognise improper integrals as a mathematical tool with multiple applications inside and outside mathematics. **Design:** Qualitative action research. **Setting and participants:** University engineering students. **Data collection and analysis:** A proposal was designed and applied to articulate mathematical modelling by using software, where a sequence of ten selected situations involving this type of integrals was experimented and solved. **Results:** The importance of modelling as a didactic-dynamic resource is highlighted because it helps students to reach and understand real situations involving improper integrals in different contexts. **Conclusions:** Despite the numerous errors detected in the students, this strategy made it possible to demonstrate that they developed mathematical competencies, which was manifested in the progress of advanced mathematical thinking skills.

Keywords: Improper integrals, Mathematical modelling, Engineering students.

Modelagem de integrais impróprios, um estudo de caso

RESUMO

Contexto: os estudantes universitários mostram pouca clareza na aplicação de conteúdos relacionados com integrais impróprios devido a uma falta de significado que não lhes permite fazer uma ligação com o ambiente e com os problemas da vida quotidiana. **Objetivo:** reconhecer os integrais impróprios como uma ferramenta matemática com múltiplas aplicações dentro e fora da matemática. **Concepção**: investigação de ação qualitativa. **Cenário e participantes**: estudantes universitários de

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engenharia. **Recolha e análise de dados**: foi concebida e aplicada uma proposta que procura articular a modelação matemática com a utilização de software, onde foi experimentada e resolvida uma sequência de dez situações selecionadas envolvendo este tipo de integrais. **Resultados**: a importância da modelação como recurso didático-dinâmico é realçada porque ajuda os estudantes a alcançar e compreender situações reais que envolvem integrais impróprios em diferentes contextos. **Conclusões:** Apesar dos numerosos erros detectados nos estudantes, esta estratégia permitiu demonstrar o desenvolvimento de competências matemáticas nos jovens, manifestado no progresso de competências avançadas de pensamento matemático.

Palavras-chave: Integrais impróprios, Modelagem matemática, Estudantes de engenharia.

Modelización de integrales impropias, un estudio de caso

RESUMEN

Contexto: se evidencia en los estudiantes universitarios poca claridad en la aplicación de contenidos relacionados con integrales impropias debido a una ausencia de significado que no les permite hacer conexión con el entorno y con problemas de la vida cotidiana. **Objetivo**: reconocer a las integrales impropias como una herramienta matemática con múltiples aplicaciones dentro y fuera de las matemáticas. **Diseño**: investigación-acción de enfoque cualitativo, **Entorno y participantes**: estudiantes universitarios de ingeniería. **Recogida y análisis de datos:** se diseñó y aplicó una propuesta que busca articular la modelización matemática con el uso de software, donde se experimentó y resolvió una secuencia de diez situaciones seleccionadas que involucran este tipo de integrales. **Resultados**: Se destaca la importancia de la modelización como recurso didáctico-dinámico porque ayuda a los estudiantes a alcanzar y comprender situaciones reales que involucran integrales impropias en diferentes contextos. **Conclusiones:** A pesar de los numerosos errores detectados en los estudiantes, esta estrategia permitió evidenciar desarrollo de competencias matemáticas en los jóvenes, manifiestas en progreso de habilidades del pensamiento matemático avanzado.

Palabras clave: Integrales impropias, Modelización matemática, Estudiantes de ingeniería.

INTRODUCTION

In university teaching practice, students who take the Calculus II course present some difficulties. Poor clarity when applying content for solving problems shows a lack of connection with the environment and daily situations

that involve the learning of improper integrals,[1] important concepts for students who study mathematics or engineering due to their several applications, including: the calculation of probabilities to define functional rules, calculation of Fourier and Laplace transforms, and physical calculations, such as work and energy, among others.

This work seeks to provide elements that help solve the need to show and manipulate the application of mathematical content in specific problems of professional work. To this end, ten specific engineering problem situations involving the use of this type of integrals were selected and applied. For reasons of space, here we present one of them, chosen because it offers elements of analysis that help enrich the teaching and learning processes of this mathematical object, which has been scarcely discussed in the existing literature. From this selection, we described a mathematical modelling proposal, in which the teacher experiences and solves the selected situation, based on an adaptation of the Blum & Borromeo-Ferri proposal (2009), visualising it as a research instrument and at the same time as a didactic strategy. This work was carried out in two stages, and the use of a technological tool was incorporated as a strategy.

Among the results, we highlight the importance of using mathematical modelling as a didactic, dynamic-methodological resource, which helps students to understand real situations and relate them to feasible problems of mathematising situations of both intra and extra mathematical character. This strategy provided students with the possibility of taking a stance in relation to the objects studied, evidenced in the development of skills typical of advanced mathematical thinking, such as: abstracting, representing, conceptualising, inducing, and visualising different problem situations, which allowed them to synthesise, define, demonstrate, formalise, and generalise. In other words, it made it possible to demonstrate the development of competencies typical of advanced mathematics.

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[1]Defined integral that covers an unrestricted area; when one or both integration limits are infinite or when the integrant considers a function with a finite number of discontinuities in the interval in question.

BACKGROUND

We present this section from two perspectives: mathematical modelling and advanced works with integrals, seeking to focus interest on improper integrals.

Regarding Mathematical Modeling

Blomhøj (2004) argues that modelling as a didactic strategy effectively brings students closer to the conceptual and algorithmic dimensions of calculus, allowing them to connect the concepts of Integral Calculus, the algorithms that characterise it and that are present in their professional work. Meanwhile, Villalobos, Brenes, and Mora (2012) show some advantages offered by mathematical modelling, since they allow representing a problem, making decisions, generating and verifying hypotheses, making predictions and giving interpretations in a specific context, favouring learning by discovery and by cooperation, and developing positive skills and attitudes related to decision-making and peer interaction. Mateus-Nieves (2022) considers that learning mathematics provides cognitive support to student conceptualisations by placing this science in culture to describe and understand daily life situations from university mathematics. The author highlights the importance of studying topological invariants because they allow finding differences and similarities in closed three-dimensional trajectories, elements that make up the structure of a surface.

Related to Integrals

Alanis and Soto (2012) affirm that traditional calculus teaching does not allow students to recognise, understand, or apply the concept of integrals in contexts that have not been studied in class. Thompson, Byerley, and Hatfield (2013) propose a conceptual approach to calculus using technological resources (specifically simulations), where the teaching of the integral is not limited to the calculation of the area demarcated by a curve, as is traditionally done, but the activities are carried out within contexts that respond to specific situations.

Mateus-Nieves (2016) performs a didactic analysis on how three groups of university students learn to use the method of integration by parts (MIP). He identifies students' difficulties in recognising the type of integral involved in the situation posed. Shows how students make mechanistic use of the MIP algorithm, without understanding it, through repetitive exercises.

Emphasises that it is possible for students to solidly understand and apply this work with integrals, if they understand them, that is, if they realise the relationship, they maintain the relational structure of the problems to which they apply. In 2019 this same author (Mateus-Nieves, 2019) presents an extension of the 2016 research, with a study for the integral from three dimensions: epistemic, to address the historical genesis of the concept; cognitive, to consider the difficulties that students present when facing problematic situations that involve the concept, and didactic, where it proposes problem situations that aim to enrich the field of social practices that are addressed with the concept. From this study, it indicates that the mathematical notions for the integral have a high level of abstraction, fundamental in the development of advanced mathematics and difficult for students to learn, therefore, a study that takes into account the logical conditions involved in the process of constitution of this mathematical object, contributes to a better understanding of it and its possible articulation.

Mateus-Nieves and Hernández (2020) present a study conducted with three groups of university students learning Integral Calculus. First, the researchers redesign the curricular structure of the Calculus II course, articulating the epistemic, cognitive, and didactic dimensions for the various meanings identified for the integral, seeking for students to make sense of what they call partial meanings for the integral in their work. Next, they show the integral as a problem-solving tool, where students reach an abstraction level that is fundamental for the development of advanced mathematics. Finally, they identify in the teacher and students the implicit perception of a culture where learning to say what the integral is and represent it geometrically without understanding it is enough to get a passing grade (ratifying the presence of a purely formal-mechanistic approach).

Mateus-Nieves (2020) presents a reflection on work with improper integrals where he emphasises that traditional teaching does not lead students to acquire the ability to understand that there is a mathematical object called "integral", which, in turn, is unitary and systemic, constituted by several meanings (types of integrals), and that can be used in different situations of an intra and extra mathematical nature. Mateus-Nieves (2021) indicates that traditional teaching is maintained with a mechanistic approach that evaluates students in such a way that they only apply an algorithm iteratively without understanding what they are doing or the benefits of using this tool. This type of teaching suggests that the important thing is to master the procedures to solve exercises or memorise definitions and understand the demonstration of

theorems (formal approach), leaving aside the usefulness and wealth of this mathematical entity for problem-solving.

THEORETICAL FRAMEWORK

This work is based on mathematical modelling articulated with the teaching of integral calculus mediated by mathematical software.

Mathematical modelling

We adopted the theoretical position of Blum & Borromeo-Ferri (2009), as the process of constructing a model directed from a real situation to a mathematical model. Mateus-Nieves (2022, in press) shows how to materialise this process: he considers the modelling cycle proposed by Blum & Leiß (2007) in terms of seven transitions: 1) the student must understand the problem or the proposed situation, that is, he/she must construct a model of the situation (transitions: understanding and construction); 2) the situation must be simplified, structured, and precise (transitions: simplification and structuring), giving rise to a real model of it; 3) transition: mathematisation, transforms the real model of the situation into a mathematical model; 4) transition: mathematical work, the implementation of the modelling must produce mathematical results; that 5) must be interpreted in the real world (transition: interpretation); 6) transition: validation and verification of results. Here, restarting a cycle to repeat some transition(s) may be necessary. Finally, the cycle closes when the student exposes the problem and its possible solution.

Mateus-Nieves (2022) proposes the modelling cycle from two moments: 1) of *awareness*, with the aim that the students develop intuition from identifying, describing and analysing the actions they carry out when they cover the subcompetencies drawn for the transitions: *construction and structuring* (proposed in table 1 shown in the results section). This moment begins with a given situation in the real world, which can be from an image, a text or both; then, there is a transition, equivalent to the partial understanding of the problem, which can be at an implicit and unconscious level for the modeller (the teacher). Transition to *structuring* is achieved when the students reach a mental representation of the situation, where they make decisions and filters information about the problem. Here, it depends on the mathematical thinking style of the individual since it is then that they decide how to deal with the problem in the following transitions of the modelling cycle. 2) Moment of

implementation and evaluation to articulate the transitions: *mathematisation, resolution, interpretation, validation, and exposure*. This process achieves the idealisation and simplification of the problem, making the process more conscious for the student. Real modelling shows how the model is constructed from drawings or formulas that represent the real situation posed, which depends on verbal statements, the support of external representations that must include the extra mathematical knowledge that the person possesses and that relates to the real model built.

In the transition from the proposed modelling cycle, students' difficulties and progress can be demonstrated as a way to improve teaching and learning processes. During the modelling process, the student is expected to replace the cognitive object with its mathematical image. The mathematical model combining theory with experiment is facilitated when technological tools such as the computer and different mathematical software are used. Mateus-Nieves and Hernández (2020. p. 201) affirm that "the process of constructing abstract concepts such as the integral is difficult for the student to understand", so this work assumes mathematical modelling as a didactic tool to strengthen the teaching and learning processes of this type of integral, seeking to change the formal mechanistic approach that has traditionally been implemented.

During the transition of the cycle, it is key that both the modeller (teacher) and the students recognise that extra mathematical knowledge is required for the construction of the mathematical model, given that external representations appear by drawings or formulas, where the statements must be at the mathematical level, which implies doing mathematical work. The mathematical competencies of the modeller to obtain mathematical results must be interpreted, even unconsciously, to obtain real results that may be validated, discussing the correspondence between the real results and the mental representation of the situation.

Blum & Borromeo (2009) state that an individual can solve a problem by going through different cycle transitions without necessarily following the order set out above; they establish that this modelling cycle may not define a linear path of thought. The transitions of the Blum and Leiß cycle (2007) cited in Mateus-Nieves (2022) constitute reference points for the analysis of modelling strategies, which allow structuring the search and categorisation of possible errors that a student can make. The study of these strategies and errors, together with the identification of the types of thinking, provide tools for an analysis-synthesis process that allows the development of the profile and detect training needs in the participating students.

Improper integrals

Within the field of integration, improper integrals occupy a special place, the reason for the study of this document, because they represent a powerful tool when there are defined integrals where one or both integration limits are infinite or when the integrant considers a function with a finite number of discontinuities in the interval in question. The convergence of the integral occurs if the limit exists, otherwise, it is said to be divergent. In the study of the defined integral $\int_a^b f(x)dx$, we understand so far that: 1) The integration limits are finite numbers. 2) The function $f(x)$ is continuous in the interval$[a, b]$. If f is discontinuous, it must be bounded in this interval. When one of these two conditions is eliminated, the resulting integral is said to be an *improper integral*, which can be classified as first, second, or third kind. It is important to clarify to the student that any improper integral can, through a suitable change of variable, become an improper integral of the first kind. One of the topics to work with this type of integrals is the comparison criteria that we describe here for improper integrals of the first kind.

First Comparison Criterion: Let *f* and *g* be two functions $[a, +\infty) \rightarrow \mathbb{R}$, integrable in the interval $[a, \alpha], \forall \alpha \geq a$, such that there is a number $b > a$ and $0 \leq g(x) \leq f(x) \; \forall x \geq b$. If $\int_a^{+\infty} f(x)dx$ is convergent (divergent), then $\int_a^{+\infty} g(x)dx$ is convergent (divergent).

Second Comparison Criterion: Let f and g be two functions $[a, +\infty) \rightarrow \mathbb{R}$, integrable in the interval $[a, \alpha], \forall \alpha \geq a$, such that $\lim\limits_{x \to +\infty} \frac{g(x)}{f(x)} = L \in \mathbb{R}$:

- If$L \neq 0$, $\int_a^{+\infty} f(x)dx$, then $\int_a^{+\infty} g(x)dx$has the same character.

- If $L = 0$ and $\int_a^{+\infty} f(x)dx$ is convergent, then $\int_a^{+\infty} g(x)dx$ is convergent.

To use either criterion, we must have a function $f(x)$ whose improper integral character is known. When it comes to some application of the defined integral, and it is improper, as in the case of an area under the curve, it is necessary to consider that if the integral is convergent, the value of the limit that defines it will be the finite value of the "infinite area". If divergent, the area is considered to be infinite.

METHODOLOGY

This is a qualitative case study research-action, where the intervention aims to demonstrate changes in university students who study Calculus II. We assumed the position proposed in Simons (1996, p. 174), which shows techniques frequently adopted in case studies, such as observation and documentary analysis, where the case investigator can "tell the story of how innovation was experienced and interpreted on the field, from the perspective and in the words of key participants". The flexibility of the case study allows following and documenting changes in the sample, elements that allow the researcher to study the effects of innovation over time.

Sample: fifty students from three engineering careers, and two professors[2] who teach the subject in three groups. We sought to articulate mathematical modelling with the use of software as a teaching methodology for this type of integrals, because "it helps students to understand better the contexts in which they develop, supports the learning of mathematics allowing them to develop skills and appropriate attitudes towards mathematics" (Blum and Borromeo-Ferri, 2009, p. 47). We chose ten problem situations typical of engineering whose solution implies using improper integrals. The sample was organised into subgroups of three students to carry out the programmed activities in two moments: 1. awareness-raising; 2. implementation and evaluation, which allowed, from the records in the field diary, to monitor exhaustively the development achieved by each student.

The first moment of *awareness-raising* took place over three weeks. As a preparation for what we call here pre-modelling, the sample was introduced in a mathematical real-context problem situation, based on two strategies: 1) To present the contents of the program from a known mathematical model, for example, the defined integral taught as a technique to calculate an antiderivative (presentation and management of the algorithm). Conceptualise the integral defined from different meanings[3], including antiderivative of a function; area under the curve; notion of accumulation and rate of change as

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[2]Both the students and the professors in charge of teaching the subject were adults and approved to participate freely and voluntarily in the research by means of an informed consent form that each signed, always safeguarding their identity in the results that are exposed. Therefore, authors explicitly exempt Acta Scientiae from any consequences arising, including full assistance and possible compensation for any damage to any research participants, per Resolution No. 510, of April 7, 2016, of the National Health Council of Brazil.

[3]The meaning of a mathematics refers to the use given to it in the language game in which one participates. Godino (2002) indicates that the meaning (of a mathematical object) is linked to the problems in which one participates and to the actions that are taken to solve those problems.

mutually dependent processes. Objective: to allow students to interpret those meanings, articulating them to different problem situations; initially, to situations of an intra character and then to other extra mathematical situations. Each situation analysed should first be plotted in the notebook, then using GeoGebra online and other open access graphics programs chosen by the students. In this way, the students became familiar with the ten situations, visualising the graphs of the regions, identifying, relating, and expanding the meaning of the integral as a reason for change, facilitating an approach to the notion of accumulation as mutually dependent processes. 2) To implement the qualities, present in the given situation in logical-numeric algorithms (of the chosen software). The objective of this moment was to identify, describe, and analyse the actions that students perform when they face problem situations posed, and what meaning they assign to a mathematical object that requires the use of improper integrals with functions of a variable. This is to cover steps 1 and 2 of the transition of the modelling cycle proposed in Table 1.

The second moment of *implementation and evaluation* sought to mathematise the situation from the data of the context, synthesis, and return to the real context, seeking to articulate the five transition steps of the cycle proposed in Table 1. Objective: to ensure that students mathematise the proposed problem situation, interpret the possible ways of solutions, and validate if the response satisfies the needs raised in the situation. As a pedagogical strategy, each subgroup submitted a written report sharing the results achieved. Anonymously, the researchers shared each work with the general group independently of whether or not the answer found satisfied the problem request. The idea of anonymity was to make students feel confident when watching their work exposed to the general group's observations. No student felt at a disadvantage before their peers when exposing their work, of which a register was written in the field diary.

At both times, the work in subgroups sought for students to discuss, analyse, construct, and investigate possible solutions to the proposed situations and, finally, deliver a written report, where only the researchers knew to which subgroup each production corresponded. The general group should check whether the modelling proposal offered met the needs raised in the problem and whether it was sufficient or incomplete. For this manuscript, we selected one of these situations because it represents the students' ease and difficulties while developing the modelling cycle and, we believe, contributes to the teaching and learning processes of integral calculus, as shared in detail below.

RESULTS AND ANALYSIS

We sought to articulate the cycle of mathematical modelling with the use of software in both moments. We started from the premise that incorporating technology as part of the process deploys autonomy in students regarding the teacher and allows them to develop skills to search for information, fostering greater participation and interest in the class. Table 1 shows the process followed. It should be clarified that after analysing and discussing the answers offered by the subgroups with the general group, each subgroup presented a written report to support the work carried out, in which they made the required adjustments to the ten proposed situations, according to the plenary discussions on the suitability of the situations and the formalisation of the content of the program by the teacher.

Table 1

Articulation of mathematical modelling cycle. (By the author)

| Moments	Transitions Modeling Cycle	Subcompetencies from the Integral Calculus
1 awareness-raising	Partial understanding of the problem	It presents the algorithm for calculating an antiderivative (presentation and handling of the algorithm, Barrow rule). It identifies the relationship between: area under the curve and the defined integral. It represents the function on the Cartesian plane and determines the region that defines the requested area. It uses online software to identify and visualise the region's graph.
	Mental representation of the situation	It indicates the functional definition of the two-dimensional figure represented in the drawing. It depicts in a diagram all the equations (lines and curves) that represent the parts of the represented function. It constructs the graphs of the equations considering the domain of the represented function. It recognises that the defined integral of the exchange rate of an amount gives the net change of that amount. It associates those defined integrals: they describe the accumulation of a quantity, so the defined integral gives the net change of that quantity. It relates exchange rate and accumulation as clearly dependent processes
	Idealisation and simplification of the problem (Interpretation)	It indicates the area covered by the function in the variation assigned for x. It recognises the area of the region as the defined integral in the given interval. (Riemann sum). It identifies, from the program, the volume and/or surface of the solid of revolution. It relates the volume and/or area of the surface as a result of the calculation of an improper integral.

2 implementation and evaluation	**Actual model** (Mathematisation-Resolution)	It relates the surface of the three-dimensional figure to the previously calculated two-dimensional surface. It relates the measurement units for each case. It establishes the solid of revolution that represents the given situation. It builds three-dimensional figures from well-defined functions, digitised in the software of a solid of revolution. It correctly expresses and records in the software the intervals in which the function that represents the proposed problematic situation is analysed. It calculates the volume and/or surface of the solid of revolution and expresses the result using the specific measurement units. It diagrams the corresponding solid in the software.
	Validation	It expresses concretely and in a real context the results found related to the surface and volume of the solid generated. It recognises the measurement units of the surface and volume generated by the solid. It checks whether the results obtained correspond to the situation raised and meet the requested needs. It socialises the process developed throughout the experience. It consistently supports the work developed and the possible extension to other possible situations.

First moment: awareness-raising

We present the selected situation since it offers elements of analysis that contribute to the research. During mathematical pre-modelling as a strategy, students had already learned what a defined integral is and how it is calculated. They were asked to graph the function in their notebooks $f(x) = \frac{1}{x}$ and calculate the area of the region included in the interval $1 \leq x \leq 3$. Then the students had to explain how they had performed this process and give the value of the area (Figure 1a), after which they had to go to the software and perform the calculations and compare the results and decide on the advantages of using the technology (Figure 1b).

Figure 1

Production of E13, 1a. Production in the notebook; 1b production with the software
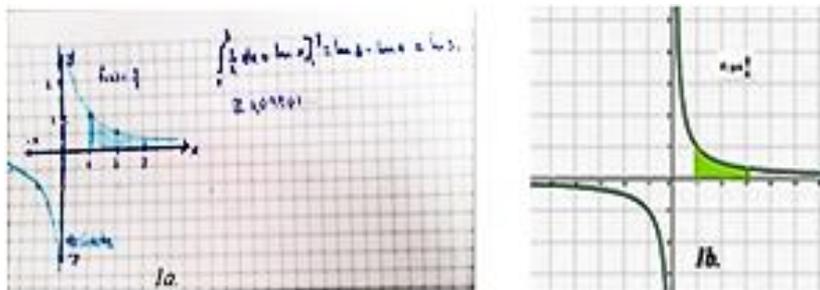
They were asked to graph and interpret what happens if we extend the integration interval between $1 \leq x \leq \infty$ (Figure 2) and express the integral. The answer obtained was: $\int_1^\infty \frac{1}{x}\, dx$, although until that moment, it had not been defined that this record corresponds to a *new integral*, called *improper*, which implies additional work to that applied to the defined integrals. We asked them: *Does this integral allow you to calculate the area of the region bounded by the curve in the given interval*? Some students mathematised using pencil, paper, and calculator, trying to find an answer. Student E3 mentioned: "*Professor, in the calculator, we got a mathematical error*, *have we done something wrong*?" to which the professor in charge asked them: *How did you do the work*? This subgroup failed to respond when another comrade, E12, said: "*We didn't do it in the notebook, we used the software, and it says that the integral is divergent. What does it mean, professor*?" (Figure 2b). E39 says: "*Professor, why does each subgroup have a different answer? [records taken from the field diary corresponding to subgroup 3]. From these productions, the mathematisation of an improper integral was formalised as a defined integral where one or both limits can contain the infinite, or as those where there are discontinuities for the function over the interval in question, seeking to articulate those meanings with the three kinds of improper integral, and, in turn, these with different problem situations.

After the concept of improper integral and the three kinds had been formalised in the activities given to the students, we asked them to expand the

concept by rotating this figure on the *x-axis* of the plane,[4] for which they had to graph the result on paper. Figure 2c shows the production of student E24. Then they had to repeat the process in the software, compare the images, discuss the situation, and think about whether it was possible to calculate the volume and the surface of that portion of solid that was formed, formalising that, in mathematics, this construction is known as "*Torricelli's trumpet*" or "*Gabriel's horn*".

Figure 2

2a. Drawing of the curve in the interval $1 \leq x \leq \infty$. 2b production of E21 in the software. 2c production of E24.
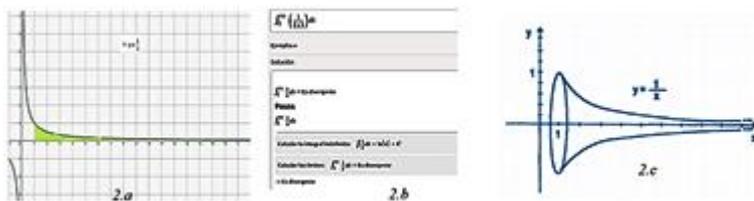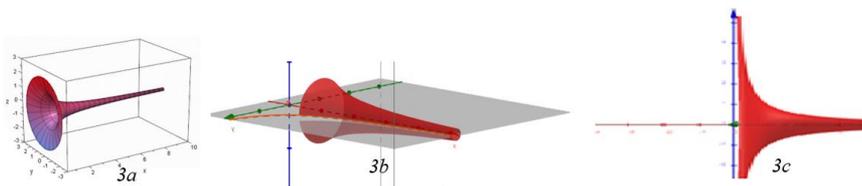

Figure 3

Layout of Gabriel's trumpet. 3rd horn of Gabriel[5]; 3b. Torricelli's trumpet[6]; 3c Torricelli's trumpet[7] when $x \in (0, \infty)$ is analysed

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[4]It was previously pointed out to them that this kind of turn in function generates a solid of revolution.
[5]Image available at https://maskupnfm.wordpress.com/2011/11/10/cuerno-de-gabriel/
[6]Solid of revolution generated by rotating around the x-axis the surface bounded by the curve $y = \frac{1}{x}$ and the *x-axis* of the plane. Source: https://www.geogebra.org/m/th9kdb8c
[7]To better understand these difficulties, it is recommended to deepen the epistemological and cognitive aspects of these concepts.

Student E19's subgroup used GeoGebra online applets and provided the images listed in Figure 3 to answer the questions. This 3D visualisation allowed them to observe the variation in the size of the solid as x approached infinity or retreated to 1 and the formation of the solid as y varied.

From this situation taken from the registers of the field diary corresponding to subgroup 14, we can analyse: it is a problem framed in a purely mathematical context. It is a function in an open interval on the right. Students are asked to try to calculate the area (the integral) using graphical and numerical methods. From the results delivered, we can differentiate two types of students: those of the first group are more confident in their knowledge and risk doing the work using their notebook and calculator, and those of the other group use the software directly. We observed that both types of students used graphical and algebraic registers of semiotic representation. All perform the graphs well, located on the axis in the requested interval, then the construction of the solid of revolution. This shows adequate handling of instrumental procedures and graphic and symbolic representations. But in calculating the area of the region, they expected approximate values of the total area as if it were a "defined integral." The students who used the software, after seeing the answer "*the integral diverges*" (Figure 2b), got confused because they expected a number. The students who did their work in the notebook made a mistake when they took the extremes of the integration interval at the points of discontinuity of the function, in this case ∞, and evaluated them in the same way as in Figure 1a. This error could be related to at least two central concepts of the calculus: *continuity* and *completeness* of the real numbers not treated in this work.

We observed in the students that the discontinuity of the functions leads them to make a series of errors in which they apply the graphical, numerical and algebraic methods as routine procedures without asking themselves the best way to solve the problem. The algebraic resolution is extrapolated directly from the one applied to the defined integral. E18 mentioned: "*What we did was to apply Barrow's rule, then we set the limit, and it gave us a value*". On the other hand, the students who directly used the software were not able to establish a connection between the results obtained with the algebraic and graphic registers, did not consistently identify the information provided by the problem, nor did they coherently coordinate the different records used with the data provided by the software. They solved the task correctly, but they did not know how to interpret it because they did not know the process followed (the step-by-step) by the software. Some students who had flunked and were taking the course again identified it as an improper integral, instead of the ordinary

Riemann integral; however, they failed to interpret because they find it difficult to express concretely and in a real context the results related to the surface and the volume of the solid generated. For these students, it is not yet clear that applying an improper integral where one of its boundaries is infinite is related to evaluating the integral of a probability density function[8].

Subsequently, based on this specific situation, the existence of improper integrals was experienced, resolved and formalised, determining the conditions necessary to classify them as first, second or third species. The third species were formalised from the question: *Is it possible to consider Torricelli's trumpet when $x \in (0, \infty)$?* (Figure 3c). From there, this type of integral was defined as the sum $\int_0^\infty \frac{1}{x} dx = \int_0^a \frac{1}{x} dx + \int_a^\infty \frac{1}{x} dx, \ a \in (0, \infty)$.

Based on this previous modelling, we proposed other situations for students to outline a possible mathematisation to find some kind of feasible solution. The sequence of programmed activities was designed so that the students could justify the processes followed. To fix the concepts, we used analogue examples that could be related to their daily life, for example, calculating currents, capacitances, current loading and unloading times, and determining the forces exerted by a fluid at the end of a tank, among others. We found that at that time, the students were prepared to address the necessary steps involved in the modelling cycle methodology.

Second moment: implementation and evaluation

We seek to articulate mathematics vs validation (Real Model), responding to the objectives in Table 1. For this moment, we resume the work advanced by the students when they built on paper the graph of the function $\int_1^3 \frac{1}{x} dx$, then in $1 \le x \le \infty$. Subsequently, it was formalised that when this figure rotates on the *x-axis* of the plane, a solid of revolution is generated. The students had to visualise the region and trace the three-dimensional figure in a two-dimensional drawing in their notebooks and build well-defined functions to visualise the three-dimensional figure in the software. Here, we detected that student presented the following difficulties: *structuring*; although the graph (model) built was correct, students could not identify the algebraic expression(s) that represent the region. This was observed because

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[8]This function evaluates the probability of a particular event occurring at some number in the interval [0,1].

the students did not take into account knowledge previously acquired. We identified that *the transition between graphical and algebraic registers* was hard for them. We noted that, in some cases, the equations presented did not correspond to the figure drawn. They were asked to reflect on this situation and make adjustments, an aspect that hindered their learning process. In particular, we identified that they found it challenging to calculate the volume, given that some students do not relate it to a solid of revolution, we believe, because they had not developed the ability to coordinate results and intuitions to present them formally.

When presenting weaknesses in the two aspects mentioned in the previous paragraph, we identified that interpretation failed, manifested in the students' difficulties to recognise and relate that the volume is finite, but the surface is infinite, this for the Torricelli's trumpet. Figure 4 shows student E20's production on paper; he struggles to understand the calculations made. At the bottom, he writes: *Why are area and volume different? I assumed that both diverged*" [records taken from the field diary corresponding to subgroup 9]. This allows us to show that he has implicit in his mind that the function tends to infinity, in which case the integral must be divergent. This reasoning is caused by the graphic record since the trumpet extends "infinitely". Here the researchers' interest in determining how many students realised this situation arose. We observed 16 out of 45, which allows us to infer that most show a lack of significance of the previous concepts and relate them to the graphic record; in terms of González-Martín (2002) this is an obstacle of *bonding compactness*, since it seems that the students only conceive a volume as finite if the figure is closed and limited.

Similar difficulties were evident when other situations of the ten proposals were worked on, including: a) Calculating the work done to raise an object of mass *m* to a defined height *h*; then determining the work necessary to drive it to an infinite distance from the Earth. b) Determining the force of gravity exerted by a uniform rod whose density is ρ=4 Kg/m and which occupies the entire positive part of the *x-axis* ($x \geq 0$) on a mass $m = 1 Kg$, located at the point of coordinates (-2m,0).

Figure 4

E20 Student Production
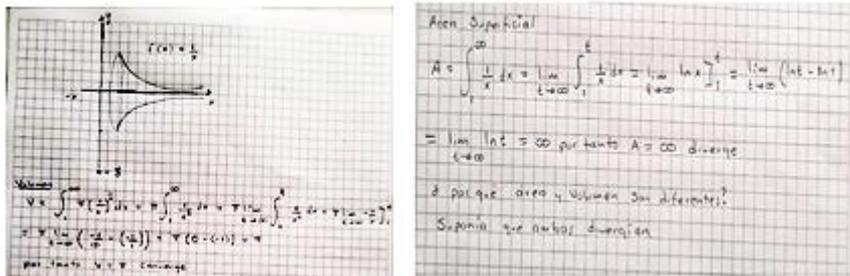
The work carried out showed the students' creativity, since each subgroup, when faced with the construction of the proposed figure, had to imagine it broken down into two-dimensional pieces. This activity allowed them to understand the importance of the continuity of a function, identify the domain and the range, and establish the difference between function and relationship, as well as the transition between the graphical representation and its corresponding equation. That is, overcoming the difficulties encountered during awareness-raising (aspects: partial understanding of the problem, mental representation of the situation and interpretation of the problem).

Progressively, we sought to integrate mathematics with other areas of knowledge, particularly physics, which allowed students to observe interest in mathematics in relation to its applicability, apprehension of mathematical concepts and development of advanced mathematical thinking skills, sharing the position raised by Mateus-Nieves & Rojas (2020, p. 69) that "in higher education, mathematics is an essential part of the learning process", "in higher education, progressive mathematisation implies the need to abstract, define, analyse, and formalise" (p. 70), and among cognitive processes with a psychological component: "represent, conceptualise, induce and visualise" (idem, p. 70).

CONCLUSIONS

Mathematical software offers students clarity in the execution and hierarchy of operations, recognising the graphs of a function as a category that articulates modelling and technology. Here we share Trouche's position (2004)

that using technological resources as a learning factor, where the electronic device becomes an instrument for plotting, calculating, and solving situations, allows the implementation of duo modelling-technology in the construction of mathematical knowledge in the classroom.

The commitment during implementation and evaluation are among the weaknesses that require more work because, although students try to interpret the results, they are not used to explaining them, making them feel insecure about the work done. Moreover, sometimes they do not trust the software as always truthful and infallible, without considering that if data is incorrectly registered, the software will produce inconsistencies, as described above. Therefore, it is necessary to recognise that the use of algebraic and graphic intuitive reasoning is related to the level of understanding of the concepts involved. Thus, students with significant conceptual gaps will hardly be able to use this reasoning.

We noted the widespread belief that the properties of a figure, in several dimensions, are maintained as this number increases or decreases. Thus, a figure of infinite area should generate an infinite volume, possibly due to the absence of coordination between records, what González & Camacho (2002) call the presence of the obstacle of homogeneous dimensions. A static conception of the boundary process as a simple algebraic operation was also evidenced, identified in the difficulties in conceptualising the calculation of the area of an infinite-looking figure.

Among the errors analysed, the lack of sense when misinterpreting the statements of a theorem or criterion and using it to solve a situation not related to the case stood out, leading to incorrect answers. For example, thinking that a function is symmetrical with respect to an axial axis, they evaluate it in an integration sub-interval and then multiply it by 2, thinking that they will get the total integral value as if the axis of symmetry was the *Y-axis* of the plane. This confirms a lack of coordination between the graphical and algebraic registers and a poor understanding of the definition of the defined integral. The fact of not having clear criteria and their use reflects also a poor connection with the meanings attributed to the integral. Therefore, we consider that a correct understanding of the concept of improper integral requires students to visualise the calculation of areas as a dynamic process that allows them to conceive the integral function and then calculate its limit, and not only think of it as an area that must always be positive.

Difficulties encountered in the development of the proposal

In one of the situations,[9] it was necessary to work on the area between curves, where the students had to propose the defined integral and solve it from the graph provided. In algebraic resolution, they extrapolated the procedure using a piecewise function at different sub-intervals, thus showing adequate handling of instrumental procedures. However, in this situation, we observed students' limited development of conceptual thinking, since they are prevented from transiting through the different semiotic representations. They find it difficult to connect the results obtained with the algebraic and graphic registers, evidencing their inability to consistently identify the information that the problem provides and the lack of coherent coordination between the different registers. They solve the task correctly with the help of the software but make mistakes on paper; the fact that the function is continuous piecewise seems to greatly influence their reasoning. Therefore, we consider that raising this type of situation requires the teacher's creativity since the evidence shows that many of those activities are not found in the textbooks proposed in the course bibliography.

There was a lack of meaning of fundamental tools for the understanding of concepts such as the use of limits, the notion of convergence, the definition of defined integral, and some minimal knowledge of sequences and series (convergence criteria), without which it will be challenging to acquire an adequate understanding of the concept of improper integral. This is an issue that remains open for future research. It confirms what Mateus-Nieves and Moreno (2021) stated on the need for more research projects to renew the teaching-learning methodologies of the basic sciences for university students, a little-discussed topic.

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[9]Students build a tunnel 2.3 km long by 15 m wide. The tunnel shape is an arc whose equation is $y = 25 \cos (\pi x/50)$. Then they must calculate the area under the assumption that the tunnel has an infinite length.

AUTHORS' SUBMISSIONS

The authors developed the research into an equitable work. Author 1 constructed the formal structure of this work as a result.

DECLARATION OF AVAILABLE DATA

The data supporting this study will be made available by the author upon reasonable request.

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