ACTA SCIENTIAE

ISSN: 2178-7727
DOI: 10.17648/acta.scientiae.5819

A Nautical Instrument Lying of the Plane in Teaching the Concepts of Chord and Arc of Circumference

Francisco Wagner Soares Oliveira [a]
Ana Carolina Costa Pereira [a]

[a] Universidade Estadual do Ceará, Programa de Pós-Graduação em Educação, Fortaleza, CE, Brasil

Received for publication 6 Sep. 2021. Accepted after review 9 Sep. 2022
Designated editor: Claudia Lisete Oliveira Groenwald

ABSTRACT

Background: This work follows in the direction of research dedicated to articulating historical instruments to teach mathematics. **Objectives**: To analyse the use of the instrument to find the altitude of the Sun as a didactic resource for teaching and learning the concepts of chord and arc of a circle in the education of undergraduate students in mathematics. **Design**: The research takes place through a university extension course that uses the theoretical and methodological foundations of the teaching guidance activity. **Setting and Participants**: The research environment is the training of students at the State University of Ceará. The training action was attended by 12 participants. The choice criterion was that they were studying for a degree in mathematics, and the selection took place in the order of enrollment in the activity. **Data collection and analysis**: From the assumptions of the teaching guidance activity, data collection was carried out with the recording of videos and audio, photographic records and students' writings, while the analysis was based on connections and interconnections on the action of the subjects, and on the conceptual nexuses. **Results**: The new instrument to find the altitude of the Sun has didactic potential to work with the concepts of arc and chord of circumference because, from it, these concepts are mobilised as practice, and it is possible to see how they relate. **Conclusions**: The new instrument to find the altitude of the Sun can be explored in the interface between history and mathematics teaching as a resource to favour teaching and learning, whether in basic education or higher education.

Keywords: Circumference chord; Arc of circumference; Formation of undergraduate students in mathematics; Teaching and learning math.

Corresponding author: Francisco Wagner Soares Oliveira. Email: wagner.oliveira@aluno.uece.br

O instrumento náutico jacente no plano no ensino dos conceitos de corda e arco de circunferência

RESUMO

Contexto: Este trabalho segue na direção de pesquisas que se dedicam a articular instrumentos históricos ao ensino de matemática. **Objetivo**: analisar o uso do instrumento jacente no plano como recurso didático para o ensino e aprendizagem dos conceitos de corda e de arco de circunferência na formação de estudantes da Licenciatura em Matemática. **Design**: a pesquisa acontece por meio de um curso de extensão universitária que tem como aporte os fundamentos teóricos metodológicos da Atividade Orientadora de Ensino. **Ambiente e participantes**: o ambiente da pesquisa é a formação de discentes da Universidade Estadual do Ceará. A ação formativa teve como público 12 participantes. O critério de escolha foi estarem cursando a licenciatura em matemática e a seleção se deu por ordem de inscrição na atividade. **Coleta e análise de dados**: a partir dos pressupostos da Atividade Orientadora de Ensino a coleta de dados foi feita com a gravação de vídeos e áudios, registros fotográficos e por escritos dos estudantes, já a análise se deu com base nas conexões e interconexões, na ação dos sujeitos e nos nexos conceituais. **Resultados**: o instrumento jacente no plano tem potencial didático para se trabalhar os conceitos de arco e corda de circunferência, pois a partir dele mobilizam-se esses conceitos de forma prática e pode-se ver como eles se relacionam. **Conclusões**: o instrumento jacente no plano pode ser explorado na interface entre história e ensino de matemática, como recurso para favorecer o ensino e aprendizagem, seja na educação básica ou ensino superior.

Palavras-chave: corda de circunferência; arco de circunferência; formação de estudantes da Licenciatura em Matemática; ensino e aprendizagem de matemática.

INTRODUCTION

The current educational scenario has been marked by increasingly accentuated demands and processes of change in the structure and design of basic education, such as the recent appreciation given by Common National Curriculum Base (Base Nacional Comum Curricular - BNCC) to the development of skills and competences (Brasil, 2017). Given this context, undergraduate students and teachers of basic education must participate in formative actions that help them to think of ways and strategies that meet the demands required by this base.

In line with these requirements, Resolution CNE/CP 02/2019, BNC - Formation points out that teachers, among other functions, should be able to "conduct the pedagogical practices of the objects of knowledge, competencies and abilities" (Brasil, 2019, p. 2). Therefore, this document reveals the valorisation given to the articulation/integration between theory and practice, learning the specific contents of each area, learning the practical dimension of knowledge, identifying pedagogical resources such as materials and artefacts to promote learning, and using different practices and resources, among others.

Taking these recommendations as a guide, especially thinking about the education of undergraduates in mathematics, this study uses Pedro Nunes' (1502-1578) instrument to find the altitude of the Sun, a resource provided by the history of mathematics to build an interface with mathematics teaching. By the interface proposal assumed here, we understand that this apparatus can favour, for example, access to the knowledge construction process, for students to observe how different mathematical concepts are related and for students to use/mobilise mathematical concepts in practice (Pereira & Saito, 2019a, 2019b; Saito & Dias, 2013; Saito, 2014).

The initiative to seek to explore instruments is still anchored in the fact of understanding that this type of apparatus is not neutral; they are part of the process that permeates the construction of knowledge (Bennett, 2003; Saito, 2009, 2014, 2019; Van Helden & Hankins, 1994). In the particular case of the new instrument to find the altitude of the Sun, research outside Brazil (Canas, 2011; Leitão, 2008; Nunes, 2012) has presented some historical and contextual information about the apparatus.

To articulate this instrument at the interface between history and mathematics teaching, we sought to appropriate the existing data that contemplate the history and context of the apparatus. In this way, in national research (Oliveira, 2019, 2020; Oliveira & Pereira, 2020a), the attention paid to the study of the new instrument to find the altitude of the Sun has focused, in large part, on its incorporation in mathematics teaching and learning, preliminarily in the education of teaching-degree students.

Various geometric knowledge can be discussed in the classroom, from the new instrument, to find the altitude of the Sun, for example: the similarity of triangles (Oliveira & Pereira, 2020a), the division of the circumference into 360 parts (Oliveira & Pereira, 2020b), the parallelism between planes, congruence of triangles, tangent line, arc and circumference chord, and perpendicularity (Oliveira, 2019). Among these possibilities, due to the presence and significance in the course of the curriculum of all basic education (Brasil, 2017) and in higher education, we sought to contemplate the concepts of chord and arc of circumference.

Given this delimitation, the question that guides this study is the following: How can the new instrument, used to find the altitude of the Sun, favour the teaching of chord and circumference arc in a pre-service teachers' training environment? In this direction, the objective outlined was to analyse the new instrument to find the altitude of the Sun as a didactic resource for teaching and learning the concepts of chord and arc of a circumference in the education of pre-service mathematics teachers.

To achieve this objective, the analysis is based on the theoretical and methodological assumptions of the teaching guiding activity (Panossian et al., 2017). In the sequence, the theoretical basis is initially exposed, then the methodology is given place, then results and analyses are presented and, finally, the final considerations.

THEORETICAL BASIS

This study has as its central environment the education of mathematics degree students. To guide the eyes on it, here, to be anchored: i) in the possibility of using the new instrument to find the altitude of the Sun in the classroom; ii) in the proposal to build an interface between history and mathematics teaching; iii) and in the theoretical elements of the teaching guiding activity.

The new instrument to find the altitude of the Sun

Belonging to the Portuguese scenario of the 16th century, particularly in the context of navigation, the new instrument to find the altitude of the Sun is proposed by Pedro Nunes [1] (1502-1578) as another alternative for determining the distance of the height of the Sun above the horizon. Regarding the physical configuration of the instrument, we have (Figure 1).

This replica follows the instructions of Pedro Nunes, set out in his work *De arte atque ratione navigandi*, modern edition published in 2008, in which he describes the instructions for the construction and use of the apparatus[2]. The

[1] For information on Pedro Nunes' biography, see Leitão (2003) and Costa (1969)

[2] The description of the new instrument to find the altitude of the Sun, found in the sixth chapter of the second navigation book presented in *De arte atque ratione navigandi*, in the 1573 edition, can be viewed between pages 46 and 47. In the modern edition of this work, published in 2008, it is arranged between pages 358, 359 and 360. In the *Petri Nonii Salaciensis Opera* of 1566, excerpts from the instrument can be found on pages 71 and 72.

instrument on display (Figure 1) is composed of a square board on which an isosceles right triangle is placed and a circumference graduated in 360 pastes is written, of which four quadrants are marked. Still on the board, at the point where the 90-degree vertex of the triangle coincides with the edge of the circumference, there is a tangent line.

Figure 1

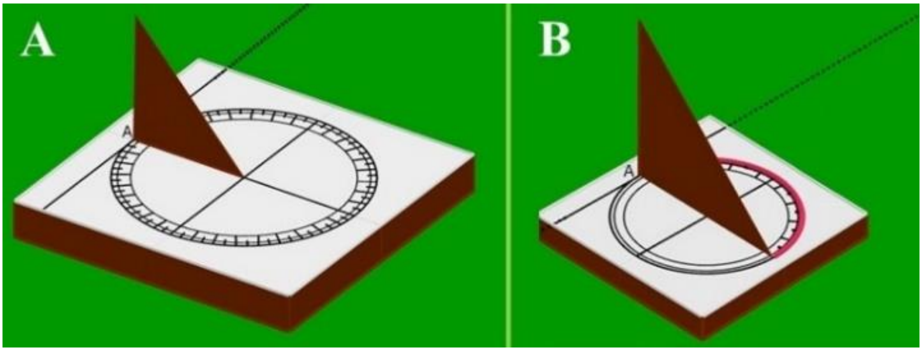
The new instrument to find the altitude of Sun. (Oliveira, 2019, p. 70)

Pedro Nunes' conception joins others already well used by men of the sea, such as the quadrant, the balhastille and the astrolabe (Silva, 1945). However, regarding the way the instrument must be positioned for use, compared to these other devices, the new instrument to find the altitude of the Sun is different from all of them, as its support board must be parallel to the horizon plane, while the others must be positioned vertically (Oliveira & Pereira, 2020c).

Regarding the physical construction of the new instrument to find the altitude of the Sun, it is worth noting that it is not limited to this configuration exposed above (Figure 1); it can assume other variants (Canas, 2011; Leitão, 2008; Oliveira, 2019)[3]. One of them refers to the apparatus with the triangle placed perpendicular to the board with two sides congruent to the diameter of the circumference. Physically, the two variations can be illustrated as follows (Figure 2).

Figure 2

Triangle with side congruent to the semi-diameter (A) and triangle with side congruent to the diameter (B)



Still on these variants, it is worth noting that in addition to the difference in size in the sides of the triangles, they also differ in that part of the circumference, more specifically the semicircumference, now divided into 90 congruent parts. We emphasised these two variants[4] since it was possible to use both, which, in a way, was responsible for stimulating discussions among the students of the mathematics degree course around the concepts of chord and arc of circumference.

The proposed interface between history and mathematics teaching

As previously mentioned, the research is developed here based on the proposal to build an interface between history and mathematics teaching, which

---

[3] On the possible configurations and variants that the new instrument to find the altitude of sun can assume, see Canas (2011), Leitão (2008) and Oliveira (2019).

[4] For details on possible the new instrument to find the altitude of sun configurations and variants, see Oliveira (2019).

has as one of its assumptions to look at history from an updated historiographical perspective.[5] (Pereira & Saito, 2018, 2019a, 2019b; Saito & Dias, 2013; Saito, 2016a, 2016b). In this direction, it is necessary to carry out two moments: the movement of thought in the formation of the mathematical concept and the context in which the mathematical concepts were developed (Saito, 2016a, 2016b; Saito & Dias, 2013). As for these moments, it is known that:

> The first is related to thinking in the formation of the mathematical concept. It is about seeking, in the historical process, the movement of thought of the apprehension of the object and, therefore, of the development of the concept. This movement, which presupposes the mathematical object in formation, allows the formation of ideas to compose the logic of the movement of thought. However, so that the logical does not prevail over the epistemological and the foundations of mathematics over mathematics itself and its applications, the search for the context of formation of these objects is prioritised in the construction of the interface, avoiding anachronisms and the overlapping of historical themes. to the purposes of teaching. Thus, the second movement refers to the context in which mathematical knowledge was developed, that is, it seeks to observe the mathematical content, method, and reasons behind the writing of the document, contextualising the time in which it was prepared and, therefore, considering all the mathematical, technical, and epistemological characteristics, as proposed by contemporary historiography (Pereira & Saito, 2018, p. 113).

In the process of building the interface, there is no pre-established order for the development of these movements, as we understand that the researcher can conduct the study based on their needs and/or conceptions. In the case where the scholar is a mathematician, the information from the historical network possibly made it possible to establish a dialogue between history and teaching. From this conversation between the areas, some questions will emerge, for example, epistemological and mathematical (conceptual), which

[5] The updated historiographical perspective consists of an investigative approach that seeks to situate knowledge in its time and space, that is, it seeks to reconstruct knowledge without introducing concepts alien to the time. It also takes into account the circumstances in which the knowledge was created (Saito, 2016a).

can present themselves as potentially didactic or pedagogical for the teaching of current mathematics (Pereira & Saito, 2019a).

In other words, the dialogue of these issues in the interface "[...] makes a new object emerge that, removed from the historical mesh, is oriented to the teaching of mathematics" (Pereira & Saito, 2019a, p. 5). Thus, the educator can propose activities for working with students to "[...] reflect the process of knowledge production that, depending on the educator's intention, may be oriented towards different teaching proposals" (Saito & Dias, 2013, p. 11)[6].

Thinking about the elaboration of activities, this interface proposal does not yet delimit a theoretical-methodological approach. We can note that some of the studies in this direction have worked under the moulds of the theory of didactic situations of Guy Brousseau (Saito, 2017), Luis Radford's (Santos, 2022) theory of objectification, and Leontiev's (1903-1979) activity theory. Taking into account the objective of this study and the fact that the lying on the plane nautical instrument is a historical artefact that has incorporated knowledge, we appropriate here the theoretical and methodological elements of the teaching guiding activity.

Theoretical Aspects of the Teaching Guiding Activity

The teaching guiding activity (TGA) is a way of organising teaching that teachers can refer to organise/guide their action so that it will favour the teaching and learning process. One of the central ideas is to take as a basis for the organisation the knowledge that has been elaborated on the human processes of knowledge construction (Moura et al., 2016).

At the heart of this model, theoretical knowledge is the main content of the activity. The appropriation of this knowledge by the students, that is, the transformation of the subject's psyche put in constant learning activity, is one of the objectives of the TGA (Moura et al., 2010; Moura et al., 2016).

The theoretical bases of TGA are the activity theory and the cultural-historical theory (Moura *et al*., 2016). In this way, we understand that it is in the light of these contributions that this way of organising teaching seeks to

[6] For details on the development of proposals in this direction, see some studies already carried out, examples of which are: Santos and Pereira (2021), Oliveira (2021), Silva and Pereira (2021), Albuquerque and Pereira (2018), Alves and Pereira (2020), and Batista and Pereira (2020).

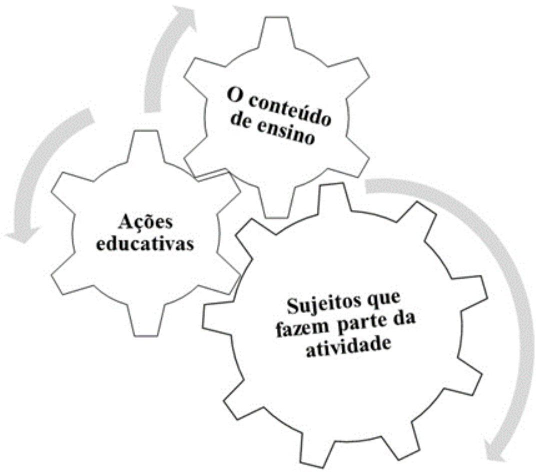
make students mobilise the concepts they already hold in thought as a way of attributing meaning to conceptual objects and re-signifying or building a new concept.

In this study, the activity theory (Leontiev, 1983, 1978) comes into play as a basis for assigning meaning to the notion of activity. On the other hand, the cultural-historical theory gains space to make it possible to understand the production/construction of knowledge and to determine whether it is appropriated by the subjects.

Under these theoretical foundations, the activity incorporates as central elements the educational actions, the teaching content, and the subjects that are part of the activity (Moura et al., 2016).

Figure 3

Core elements of the activity (Oliveira, 2019, p. 82)



This illustration seeks to point out the relationship of dependence between these elements, in which the movement of one exerts a direct influence on the others. With regard to the subjects participating in the activity, teacher(s) and student(s), it is known that they are in constant activity and are seen as bearers of knowledge, values, and affection. One of the functions of the teachers is to organise teaching to enable students to appropriate both knowledge and historical-cultural experiences (Moura et al., 2016).

From this relationship, the pedagogical activity is nourished and takes shape, moving towards a structure that privileges "[...] a need (appropriation of culture), a real motive (appropriation of historically accumulated knowledge), objectives (teaching and learning) and proposes actions that consider the objective conditions of the school institution" (Moura et al., 2010, p. 217).

The need/appropriation of culture is linked to the origin of the activity and the appropriation of certain conceptual objects (content) that must be discussed in the classroom. The motive concerns the acquisition of specific concepts of the subject to be taught. In turn, the objectives refer to educational actions, both teaching and learning, which, in a general scenario, are associated with practical operations that direct the development of a training action. In the relationship and approximation between educational actions, teaching content, and subjects, the apprehension of conceptual objects must be seen by students as a necessity to complete the activity (Moura et al., 2010; Moura et al., 2016).

METHODOLOGY

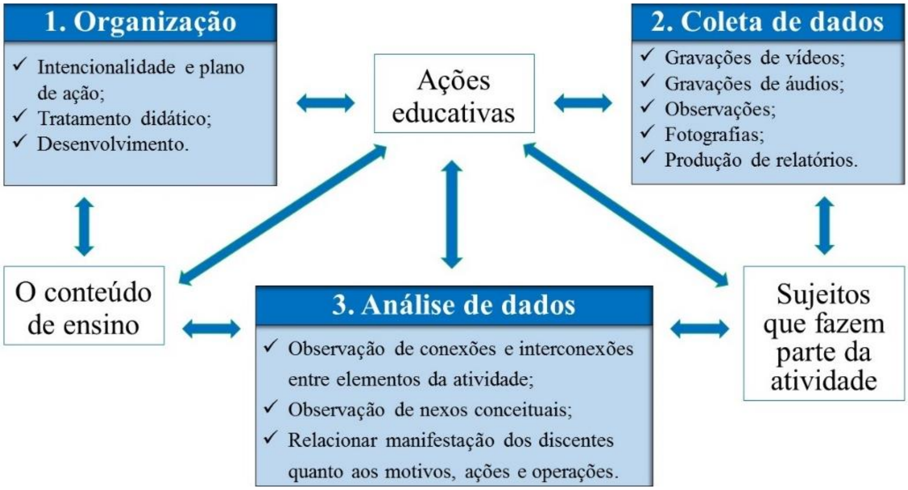
In this study, we describe a training action in which elements of the nautical instrument lying in the plan were explored with undergraduates in mathematics[7]. Here, as previously pointed out, the focus is on analysing the use of the instrument above as a didactic resource for teaching and learning the concepts of chord and arc of circumference in the education of mathematics degree students.

In this direction, the research as a whole is outlined in the light of a qualitative approach (Lüdke & André, 2013) and based on the proposal to build an interface between history and mathematics teaching. For the proposed activity, the theoretical methodological elements of the TGA are followed in the stages of data construction, application, collection, and analysis. In these terms, the course of the training action assumes the following structure (Figure 4).

[7] This research is part of the project "Some potentialities of the construction of the lying on the plane observed at the interface between history and teaching of mathematics", approved by the UECE CEP under Opinion number: 3.599.527 and CAAE number: 19561119.6.3001.5534.

Figure 4

Activity structure



The organisation of the training action is anchored in the perspective of building an interface between history and mathematics teaching (Saito & Dias, 2013). In the stage of intentionality and action plan, the objectives and intentions are in general terms. The didactic treatment refers to decoding the text (excerpts) in which Pedro Nunes describes the instrument lying in the plane, favouring the students' understanding. Development, as its name suggests, concerns the execution of the activity in practice (Saito & Dias, 2013; Saito & Pereira, 2019). The collection and analysis of data largely follow the theoretical and methodological aspects of the TGA observed in some actions already developed in light of this way of organising teaching (Panossian et al., 2017).

The training action analysed in this work was carried out in 2019, with a workload of 20 hours/classes shared between the 1st, 2nd, and 3rd of August. It took place in the format of an extension course, at the State University of Ceará (UECE), specifically at the Laboratory of Mathematics and Teaching Professor Bernardo Rodrigues Torres (LAbMAtEn/UECE) and in the roundabout square of that institution.

The participating public were 12 students of the Mathematics Degree at UECE, distributed proportionally into four groups. Here, each student received a specific codename, with only one variation occurring in terms of gender, from student 1 through student 12. Group 1 is composed of the first three students (student 1, 2, and 3), following this logic until group 4 (student 10, 11, and 12).

On the first day of the activity, the students had to try to understand the apparatus based on Pedro Nunes' description. On the second day, in addition to the excerpts from the 16th century, to encourage the study of the participants, a physical replica of the new instrument to find the altitude of the Sun was given to each group. Finally, on the third day, there was a moment in the square of the UECE roundabout with the students in which they were requested to use the apparatus.

As the focus is on mobilising the concepts of bow and string, here the discussions held by the students on the second day of the training action (August 2nd, 2019) are given. Specifically, they focus on the students' discussions about the arc of the height of the Sun, which is indicated by the shadow projected by the isosceles right triangle, which must be placed perpendicularly to the board and which, depending on the configuration of the apparatus, can assume different sizes. Different, but still congruent.

When analysing the measurement provided by the projection of the shadow of the triangle on the board for the two configurations indicated in Figure 2, the students felt the need to understand what an arc and a chord of a circumference would be. Regarding this moment, it is worth noting, again, that each group used Pedro Nunes' description of the new instrument to find the altitude of the Sun and a physical replica of the apparatus as a resource.

Subsequently, space is given both to the results collected from the recording of videos, audios, photographs, and production of reports (Figure 4) and to the analysis, which has as criteria to observe the connections and interconnections between elements of the activity and the conceptual nexuses, and still relate the manifestations of the students regarding the motives, actions, and operations (Figure 4).

Still on the analysis, under the contribution of the TGA, an isolated individual was taken as a category/minimum unit of analysis[8], defined here as

[8] The isolated refers to taking a unit from the whole for analysis. Its definition is one of the first products of the analysis (Araujo & Moraes, 2017). As the students' discussions about the arc of the height of the Sun were woven on the second day of

the moment of students' discussions about the chord and the arc of the height of the Sun. This isolate consists of two episodes[9], the first refers to the speeches in which the students present their initial understandings, and the second refers to the moment they move towards a systematisation of the concepts.

RESULTS AND ANALYSIS

Preliminarily, it is worth noting that during the study of the new instrument to find the altitude of the Sun from the instructions of Pedro Nunes and the physical replica of the apparatus, the students mobilised and possibly meant and re-signified various geometric concepts (Oliveira, 2019), for example, the concepts of arc and circumference chord elucidated here. Two excerpts by Nunes about the construction and use of the instrument contributed significantly to this fact. The first of these refers to the instruction that:

> [...] if this type of instrument has a square shape, so that the line **ak** tangent to the circle at point **a** can be drawn on it, a stylus or a rod whose shadow is projected on the line **bd** will not be necessary. Just turn the instrument until the shadow of the line **af** is projected on the line **ak**, because then the shadow of the line **ef** will *indicate the arc of the height of the Sun above the horizon* (Nunes, 2008, p. 359, italics ours).

In this regard, we have the following representation (Figure 5).

As illustrated in Figure 5 and already pointed out in the excerpt by Pedro Nunes, here, the author is talking about the possibility of using the new instrument to find the altitude of the Sun for its construction on a square board. Note also where the arc of the height of the Sun above the horizon is indicated on the apparatus, and, in this regard, it is necessary to point out that this arc is calculated "[...] from point **b** in the direction of **a**. The remaining part of the quadrant up to **a** will be the distance between the Sun and the zenith" (Nunes, 2008, p. 358).

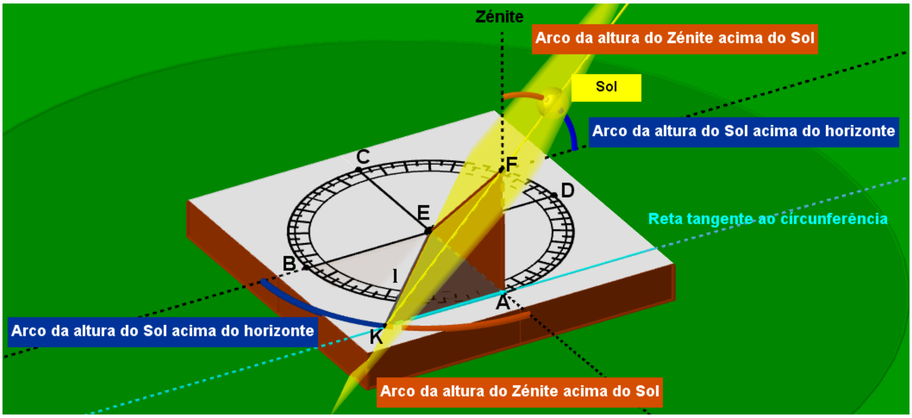
---

the activity, it is understood that they can be defined as an isolated one, because even when isolated for study, the discussions are still part of the activity as a whole.

[9] The episodes can be written or spoken phrases, gestures, and actions that constitute scenes that can reveal interdependence between the elements of a formative action (Moura, 2004, p. 276).

Figure 5

Hypothetical representation of the use of the apparatus



The second excerpt that, added to the previous one, boosted the discussion, concerns the following statement by Pedro Nunes:

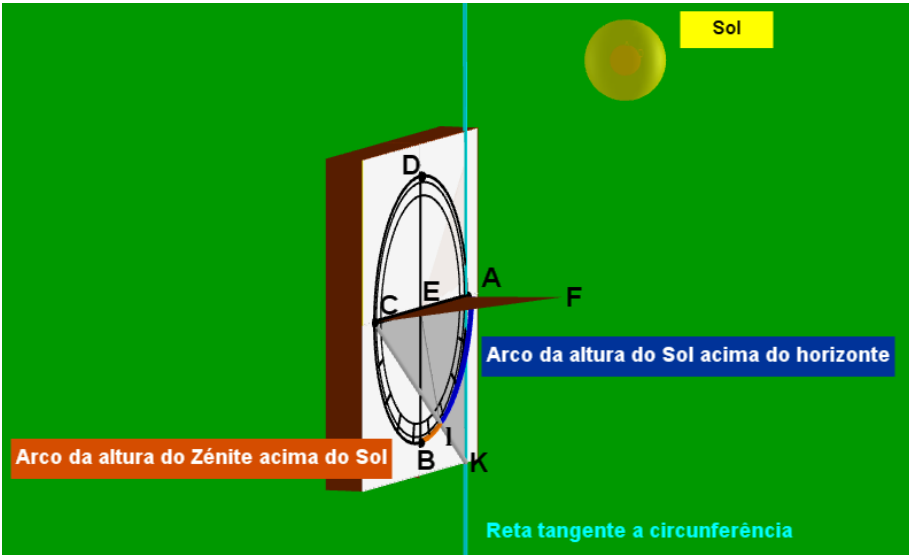
> If the sides of the triangle **fgh** are doubled so that the side **gh** is equal to the diameter **ac** and fits perfectly into it, then the semicircle **abc** can be divided into ninety equal parts, and then the degrees of the height of the Sun will be doubly bigger (Nunes, 2008, p. 359).

In light of this instruction, we have (Figure 6).

This new instrument to find the altitude of sun configuration still remains on the square board, but the difference is that it now changes the measurements of the sides of the triangle, as they must be duplicated. Constructed in this way, the apparatus must be positioned perpendicular to the horizon plane and when placed over the Sun, the shadow of the line **af** must be projected on the "[...] line **ak**, **a1** will be the arc the height of the Sun above the horizon, and the remaining **b1** will be the arc of the distance between the Sun and the zenith" (Nunes, 2008, p. 359).

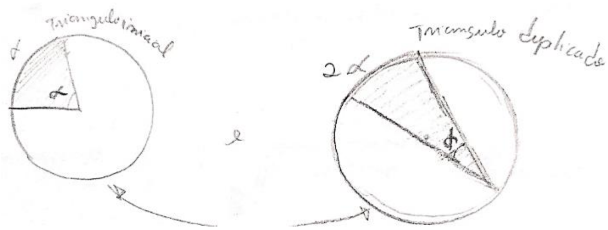
Figure 6

*Hypothetical representation of the use of the apparatus with the sides of the triangle **fgh** duplicated*



About the fact that Pedro Nunes emphasises that in this version of the instrument that "[...] the degrees of the height of the Sun will be doubly greater" (Nunes, 2008, p. 359). The students point out that:

> [...] when you are working with circumference, when you take the centre from here to here it is α, this separation is α. When you increase this here, this is going to be α divided by 2. Arc of Circumference. When you increase to an edge, you increase the angle that will form there, so that's why now, as it's coming from here to here, this arc is here now



(Student 11, 2019).

This relationship observed by the students is in line with the twentieth proposition of the third book of Euclid's Elements (Oliveira, 2019; Reis, 2003). In it, the author explains that "[...] in a circle, the angle near the centre is twice that on the circumference when the angles have the same circumference as their base" (Euclides, 2009, p. 169). In these terms, we understand that taking a circumference, particularly the same arc of that circumference, the angle at the centre is, in degrees, twice the inscribed angle.

Episode 1 - Initial understandings of the concepts of arc and circumference chord

Given these possible configurations for the new instrument to find the altitude of the Sun, mainly with regard to the size and arrangement of the isosceles right triangle on the board and also due to the change in the reading of the measurement of the height of the Sun on the circumference, some students are confused about what this arc of the height of the Sun would be. In this regard, the following discussion began with a questioning:

> *[Student 7]: "What is the arc of the Sun's height?"*
>
> *[Teacher]: "Guys what is the arc of the height of the Sun, can you help her?"*
>
> *[Student 12]: "It's the bow the shadow makes, now what it means I don't know".*
>
> *[Student 7]: "I know the shadow is the bow, but what is the bow?"*

From what can be observed in this dialogue, the student understood that in using the new instrument to find the altitude of the Sun, the arch will be indicated by the shadow projected on the board. The last speech of student 7, in this conversation, reveals that her doubt is related to the concept of arc. In view of this observation, we note that the knowledge of chord appears to the student as a need to understand the measure, a fact that reinforces what the teaching guiding activity highlights by noting that the teaching content, as well as other elements, must be understood by the student as a necessity to fulfil the proposed activity (Moura et al., 2010; Moura et al., 2016).

To promote the understanding of Student 7, some students start to present their meanings about the concept of arc, as follows:

> *[Student 4]: "Arc is a part of a circumference".*
>
> *[Student 8]: "It's a rope, no, I was going to say it's a chord".*
>
> *[Teacher]: "It's a chord, like she said?"*
>
> *[Student 8]: "And not".*
>
> *[Student 4]: "No, a chord is a line that intersects the circumference one or two points".*
>
> *[Student 8]: "No, the arc can be a chord too, it can!"*
>
> *[Student 7]: "He can!"*
>
> *[Student 11]: "But you can't say that a part of the circle is necessarily an arc".*

In this conversation, we note that it is attributed to the concept of an arc, which is a part of the circumference and can be considered a chord. The incorporation of the concept of chord in the discussion points to the existence of a relationship between the concepts. This fact is in line with what is valued in the BNCC, in which the organisation of skills privileges the explanation of how the objects of knowledge are related (Brasil, 2017).

In these terms, assuming the role as one of the actors in the educational process, as foreseen in the teaching guiding activity (Moura et al., 2010; Moura et al., 2016), the teacher who promotes student activity, in the sense of fostering discussions that make it possible both to articulate the concepts of chord and arc of circumference and assigning even more meanings to these concepts, launches the following question:

> *[Teacher]: "But can the arc be a chord"?"*
>
> *[Student 8]: "that I can remember"!*
>
> *[Student 4]: "A chord is a line that intersects the circumference at up to two points, one or two points. If it is one, it is called a tangent".*
>
> *[Student 1]: "But it is not the space between these two points that you are talking about, that is, the chord, no!"*
>
> *[Student 4]: "It's the chord!"*

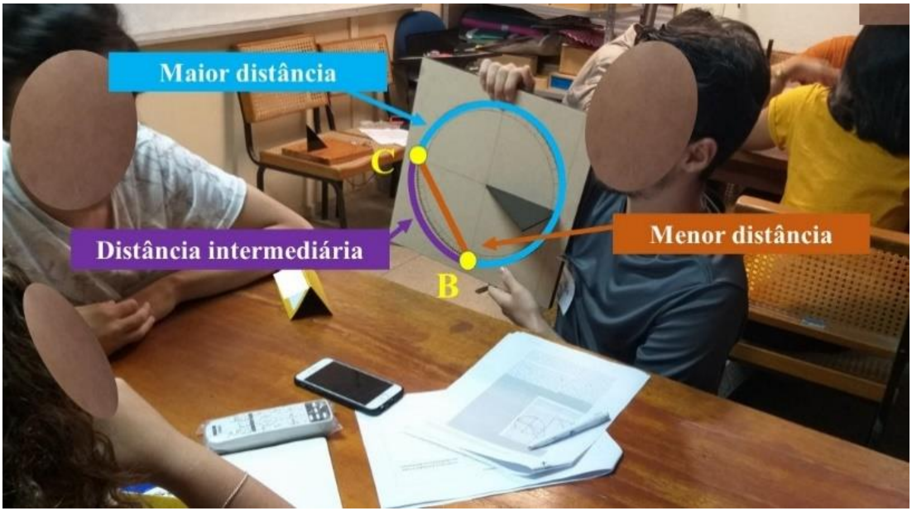
In this dialogue, the notion of a tangent line already appears as a way of justifying what would be a chord of the circumference. Insisting on the discussion, the teacher is asked: “So the chord can be the arc?” one of the students says that: “No!” (Student 11, 2019). Student 7 (2019) asks: “And what is the arc if it cannot be the chord?”. With this question, the dialogue returns to present some meanings to the concepts of arc and chord.

> *[Student 4]: “The arc is a part of a circumference”.*
>
> *[Student 5]: “what is the chord!”*
>
> *[Student 7]: “What is the chord!”*
>
> *[Student 11]: “You can’t claim it, it’s not a circumference part”.*
>
> *[Student 4]: “The chord is the intersection line”.*
>
> *[Student 7]: “The chord num is the space between the circumference and this line”.*
>
> *[Student 8]: “The chord num is the connection between two points on a circumference”.*
>
> *[Student 11]: “It is”!*
>
> *[Student 3]: “But it has to be straight”.*
>
> *[Student 11]: “It is a line connecting two points on a circumference that does not include the central point”.*
>
> *[Student 4]: “A rope is a link between two points at the shortest distance”.*
>
> *[Student 11]: “That does not include the central point”.*
>
> *[Student 4]: “The arc is a part of the circumference”.*
>
> *[Student 11]: “But it’s not necessarily a part of the circumference”.*
>
> *[Student 7]: “So I studied chord my whole semester wrong because as I understood chord, this was it”.*

In this passage, we observe that a new meaning is assigned to the notion of a circumference chord when student 4 (2019) says it “is a connection between two points at the shortest distance”. About this meaning, from the new instrument to find the altitude of the Sun, we understand that (Figure 7).

Figure 7

The chord as the shortest distance (Elaborated by the authors)



Taking points *b* and *c* as references, according to what student 4 pointed out, the chord would be that line segment, represented in orange (shortest distance). Seen this way, this meaning corroborates another already highlighted above, in which the chord is “a straight line that intersects the circumference at up to two points, one or two points. If it is one, it is called a tangent” (Student 4, 2019). In these terms, one of the students points out that: “The arc will be the distance between two points on the circumference, covering the circumference” (Student 11, 2019). Thus, the intermediate and the longest distances between points **b** and **c** would be considered arcs of the circumference.

Still on the previous dialogue, it appears at the end that Student 7 declares that she misunderstood the chord throughout the semester. In her conception, already exposed in this previous conversation, she says that: “t=The chord is the space between the circle and this line” (Student 7, 2019). This student’s statement indicates that, possibly from this study, with the instrument lying on the plane, she had the opportunity to reconfigure this concept.

Episode 2 – Journey towards a systematisation of concepts

In order to move towards a systematisation of the chord concept, the following question was raised:

> *[Teacher]: "Is there a definition of a chord?"*
>
> *[Student 4]: "It is a line segment inside the circumference that connects two points".*
>
> *[Student 11]: "That does not include the central point, which would be the diameter".*
>
> *[Student 3]: "It can include".*
>
> *[Student 4]: "It can include, if it includes, is the diameter, the diameter is the largest chord".*
>
> *[Student 11]: "Depending on the book he differentiates, there are books that differentiate the diameter of the chord, he states that the chord is the distance of points that do not include the central point".*
>
> *[Student 3]: "So, in this case, it will be like the natural numbers, there is a book that includes zero and others do not".*
>
> *[Student 11[: "Yes, it's a question of the author, but there are books that consider that the central point does not have to be included, because if you include it, it becomes diameter. Then you put it as the diameter. What diameter will be the distance of two points on the circumference that includes the center".*
>
> *[Student 4]: "The diameter is the largest chord of the circumference".*
>
> *[Student 4]: "Yes, but it depends on the author, some authors do not consider it".*

In this dialogue, it appears that to present meanings to the concept of chord, the students still bring the notion of diameter to the discussion. This fact also points to the relationship that exists between concepts when one is going to assign meaning to an object and thus construct or re-signify a mathematical concept.

With regard to the definition of chord systematised by the students in this conversation, it is worth noting that it is in accordance with what is found in basic education mathematics books, in which "the chord is the segment with ends at two points on the circumference" (Silveira & Marques, 2013, p. 285).

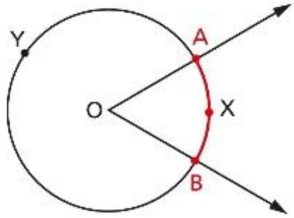
About this previous dialogue, it is worth mentioning that the students pointed out that, as well as zero, which may or may not be considered a natural number, the same occurs with the fact of taking or not taking the diameter as a chord. This discussion points to a certain maturity of students regarding the definition/criteria for assigning meaning to specific mathematical objects.

Now, thinking about systematising the meanings attributed to the concept of arc, the teacher asked:

> *[Teacher]: "What is the arc for you, in the face of discussions?"*
>
> *[Student 7]: "The arc for me is a part of the circumference, the outer part, the outline of the circumference".*
>
> *[Student 12]: "The contour of the circular sector, right!"*
>
> *[Student 11]: "A part of the length, which when you put the length you don't include the area".*
>
> *[...]*
>
> *[Student 4]: "A chord is a line segment connecting two points on the circumference. Arc is a connection between two points on the circumference".*
>
> *[Student 8]: "Now it makes sense!"*

The definition, pointed out by the students, is in line with what can be observed in mathematics books. In one of them, specifically in the third volume of the collection of Fundamentals of elementary mathematics, which is well known and worked on in teaching degree courses, we have:

> Considering a circumference with centre O and a central ring AÔB, with A and B being points that belong to the sides of the angle and to the circumference. The circumference is divided into two parts, each of which is an **arc of a circumference**: arc of a circumference $\widehat{AXB}$ and arc of a circumference $\widehat{AYB}$ A and B are the ends of the arc



(Iezzi, 2013, p. 24).

In the analysis of this dialogue and the others, we observe that the students possibly reconfigure the concepts of chord and arc of circumference. There is talk of this possible process of internalisation because the students' individual activity (intrapsychic relationship) takes shape from the collective activity (interpsychic relationship), which occurs through the interaction and dialogue between students and between students and the teacher (Vigotski, 2001). This possibility corroborates what was verified during the conversations, in which there are expressions such as "I thought it was" (Student 8, 2019), an initial intrapsychic activity. At the end, the same student states: "Now it makes sense", posterior intrapsychic activity, constituted from the interaction and mediation with the other students. This speech by student 8 points to a qualitative and significant leap in students' understanding. Another piece of evidence that points to possible learning refers to the following writing in the daily evaluation sheet[10] (Figure 8):

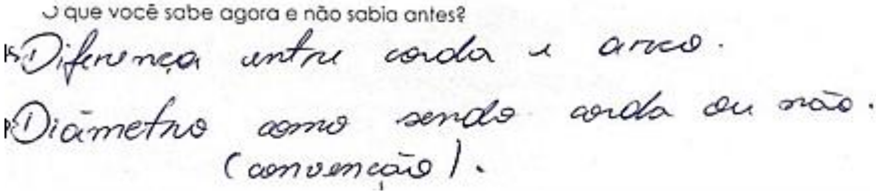
With this answer, we note that the discussions potentiated the learning of the bow and string concepts. One of the justifications for this understanding is that the students were in constant activity during the formative action, a fact that, in the light of the activity theory, is responsible for generating psychological development (Rigon et al., 2016). In figure 8, it is also observed that the student has come to understand that determining the diameter as a chord or not is just a convention, which can vary from author to author, with some pointing to the diameter as being the largest chord of a circumference, while others do not take the diameter as a chord. Regarding these points highlighted in the student's writing (Figure 8), we also understand that they approach an action of self-assessment and regulation (Davidov, 1987), which added to the

[10] The daily evaluation form is part of the report that students had to deliver at the end of each meeting.

interactions throughout the training action point to the appropriation of concepts.

Figure 8

Writing by one of the students at the end of the activity (authors' collection)



O que você sabe agora e não sabia antes?

1) Diferença entre corda e arco.

1) Diâmetro como sendo corda ou não. (convenção).

In the new instrument to find the altitude of the Sun, it is worth noting again that these discussions of arc and chord were elucidated from the possibility of using two variants of the apparatus, one with the isosceles right triangle with two of its sides congruent to the semi-diameter and the other with the sides of this triangle duplicated. Given these possibilities, these concepts are summarised in the apparatus as follows (Figure 9):

Figure 9

Instrument with a triangle with a side congruent to the semi-diameter (A) and with a triangle with duplicated sides (B)

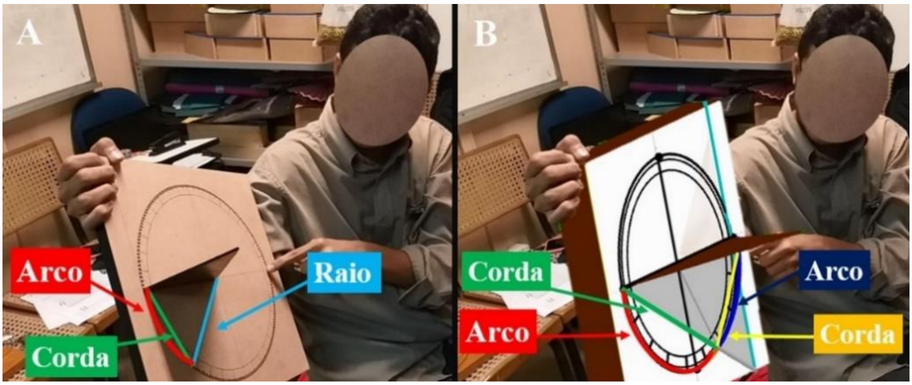


Figure 9 (A) shows the situation of using the new instrument to find the altitude of the Sun for the case in which the sides of the isosceles right triangle are congruent to the semi-diameter of the circumference. As can be seen, the shadow cast by the triangle indicates an arc on the limb of the apparatus (length in red), and from the ends of this arc, a chord can be drawn (length in green) if there is also a segment that leaves the centre of the circumference and touches it on its edge (radius). In the case where the sides of the triangle are duplicated (Figure 8B), two arcs are highlighted, one in red and one in blue, and two chords, one in green and the other in yellow.

Comparing the mobilisation of the concepts of chord and arc in the two configurations of the apparatus, we see in Figure 8A that the shadow projected by the hypotenuse of the triangle is the radius, whereas in Figure 8B, this shadow is a chord. It is changes in this sense that potentiated the students' discussions, since understanding these concepts became a necessity to understand the measure provided.

CONCLUSIONS

In view of the analysis made in this study, what can be said about: how can the use of the new instrument to find the altitude of the Sun in the pre-service teachers' training environment favour the teaching of chord and circumference arc? The research shows that the apparatus allied to the game of interactions between the teaching content (chord and arc of circumference), the educational actions, and the activity of the teacher and students can help them to understand how the concepts of chord and arc are related and to give them even more meaning.

By observing how the concepts of chord and arc of a circumference are related, students can add even more meanings to them. Besides elucidating the notion of arc and chord, to understand them, the concepts of tangent line and diameter were also mobilised in the discussion.

The new instrument to find the altitude of the Sun can also favour the teaching of chord and arc, as it encourages the mobilisation of theoretical concepts in a practical way. We understand that this potentiality can collaborate so that the students can attribute even more meaning to the concepts. It is also necessary to point out the constant negotiation of meanings driven by the study of the apparatus since it was also paramount to give new meaning to the concepts in question.

From these results, we understand that the instrument lying in the plan can be a resource explored by the mathematics teacher in the teaching degree classroom. For future research, we intend to analyse the potential of the apparatus both for working with other mathematical concepts and for its incorporation in activities aimed at students of basic education.

AUTHORSHIP CONTRIBUTION STATEMENT

FWSO and ACCP participated in the entire process of this research, which ranged from the preparation of the mini-course proposal, definition of the theoretical and methodological support, and collection and assessment exams. The writing of this article is also the production of the collective work of these authors. In these terms, both approve the final version of the work.

DATA SHARING STATEMENT

The authors agree that data supporting the results of this study are available upon reasonable request, at the authors’ discretion.

REFERENCES

Albuquerque, S. M. de, & Pereira, A. C. C. (2018). A divisão por unidades pelo método da diferença: uma proposta de uso do ábaco de Gerbert (976). *Revista de Produção Discente em Educação Matemática*, *7*(2), 73-88.

Alves, V. B., & Pereira, A. C. C. (2020). Seno, cosseno e tangente: uma atividade com os círculos de proporção de William Oughtred (1633) na formação de professores de matemática. *Amazônia: Revista de Educação em Ciências e Matemáticas*, *16*(35), 74-88. http://dx.doi.org/10.18542/amazrecm.v16i35.8275

Araujo, E. S. A., & Moraes, S. P. G. de. (2017). Dos princípios da pesquisa em educação como atividade. In: M. O. de Moura (Org). *Educação escolar e pesquisa na teoria histórico-cultural*. Loyola.

Batista, A. N. de S., & Pereira, A. C. C. (2020). A balhestilha (1603) como um instrumento matemático para o estudo de medidas na formação de professores de matemática. *Acta Scientiarum. Education*, *43*(1), 129-150. https://doi.org/10.4025/actascieduc.v43i1.48188

Bennett, J. A. (2003). Knowing and doing in the sixteenth century: what were instruments for? *British Journal for the History of Science*, *36*(2), 129-150. https://doi.org/10.1017/S000708740300503X

Brasil, Ministério da Educação (2017). *Base Nacional Comum Curricular (BNCC). Educação é a Base*. MEC/CONSED/UNDIME.

Brasil. Ministério da Educação (2019). *Resolução CNE/CP nº 2, de 20 de dezembro de 2019. Define as Diretrizes Curriculares Nacionais para a Formação Inicial de Professores para a Educação Básica e institui a Base Nacional Comum para a Formação Inicial de Professores da Educação Básica (BNC-Formação).* http://portal.mec.gov.br/docman/dezembro-2019-pdf/135951-rcp002-19/file

Canas, A. J. D. C. (2011). *A obra náutica de João Baptista Lavanha (c. 1550 – 1624)*. Tese de Doutorado, Universidade de Lisboa Faculdade de Letras Departamento de História, Lisboa. http://repositorio.ul.pt/handle/10451/6140

Costa, A. F. da. (1969). *Pedro Nunes (1502-1578)*. Agência Geral do Ultramar.

Davídov, V. V. (1987). Análisis de los princípios didáticos de la escuela tradicional y posibles princípios de enseñanza em el futuro próximo. In M. Shuare. *La Psicología evolutiva y pedagógica en la URSS:* Antologia (pp. 143-142). Progreso.

Euclides. (2009). *Os Elementos*. Editora UNESP.

Iezzi, G. (2013). *Fundamentos de matemática elementar, 7: trigonometria*. Atual.

Leitão, H. (2008). Anotações ao De arte atque ratione nauigandi. In P. Nunes. *Obras* (4.vol., pp. 515-794). Academia das Ciências de Lisboa, Fundação Calouste Gulbenkian.

Leitão, H. (2003). Para uma biografia de Pedro Nunes: o surgimento de um matemático, 1502-1542. *Cadernos de Estudos Sefarditas*, *3*(1), 45-82.

Leontiev, A. (1978). *O desenvolvimento do psiquismo*. Horizonte.

Leontiev, A. (1983). *Atividad, conciencia, personalidad*. Pueblo y Educación.

Lüdke, M., & André, M. E. D. A. (2013). *Pesquisa em educação: abordagens qualitativas*. E.P.U.

Moura, M. O., Araújo, E. S., Moretti, V. D., Panossian, M. L. & Ribeiro, F. D. (2010). Atividade orientadora de ensino: unidade entre ensino e aprendizagem. *Rev. Diálogo Educ.*, *10*(29), 205-229. http://dx.doi.org/10.7213/rde.v10i29.3094

Moura, M. O., Araújo, E. S., Souza, F. D., Panossian, M. L., & Moretti, V. D. (2016). A Atividade Orientadora de Ensino como Unidade entre Ensino e Aprendizagem. In: Moura, M. O. (Org). *A atividade pedagógica na teoria histórico-cultural* (pp. 93-126). Autores associados.

Moura, M. O. (2004). Pesquisa colaborativa: um foco na ação formadora. In R. L. L. Barbosa (Org.). Trajetórias e perspectivas da formação de educadores (pp. 257-284). Editora UNESP.

Nunes, P. J. A. (2012) *Os instrumentos náuticos na obra de Pedro Nunes* [Dissertação de Mestrado, Curso de Mestrado de História Marítima, Universidade de Lisboa Faculdade de Letras Departamento de História]. Lisboa. http://repositorio.ul.pt/handle/10451/8968

Nunes, P. (2008). *Obras, Vol. IV: De Arte Atque Ratione Navigandi*. Fundação Calouste Gulbenkian.

Oliveira, F. W. S., & Pereira, A. C. C. (2020b). Indícios do Costume Relacionado a Divisão da Circunferência em Seus 360 Graus presente na Fabricação do Instrumento Jacente no Plano de Pedro Nunes. *Revista Brasileira de História da Matemática*, *20*(39), 35-49. http://dx.doi.org/10.47976/RBHM2020v20n3935-49

Oliveira, F. W. S., & Pereira, A. C. C. (2020c). Sobre a navegação portuguesa do século XVI à luz do instrumento jacente no plano. *Boletim da Sociedade Portuguesa de Matemática*.

Oliveira. F. W. S., & Pereira, A. C. C. (2020a). Uma proposta de atividade com o instrumento jacente no plano para o nono ano do ensino fundamental com foco na semelhança de triângulos. *Revista história da matemática para professores*, *6*(2), 20-27.

Oliveira, F. W. S. (2019). *Sobre os conhecimentos geométricos incorporados na construção e no uso do instrumento jacente no plano de Pedro Nunes (1502-1578) na formação do professor de matemática* [Dissertação de Mestrado, Instituto Federação de Educação, Ciência e Tecnologia do Ceará]. Fortaleza. https://sucupira.capes.gov.br/sucupira/public/consultas/coleta/trabalho

Conclusao/viewTrabalhoConclusao.jsf?popup=true&id_trabalho=8250117

Oliveira, F. W. S. (2020) Um primeiro olhar sobre a reconstrução do instrumento jacente no plano de Pedro Nunes na formação do professor de matemática. *Boletim Cearense de Educação e História da Matemática*, *7*(20), 67–79, 2020. http://dx.doi.org/10.30938/bocehm.v7i20.2868

Oliveira, G. P. (2021). Um primeiro olhar de aspectos gerais do tratado a arte de navegar (1606) de Simão d' Oliveira. In: *Anais do XIV Seminário Nacional de História da Matemática*. Universidade Federal do Triângulo Mineiro, Uberaba-Minas Gerais. www.even3.com.br/Anais/xivsnhm/326492-UM-PRIMEIRO-OLHAR-DE-ASPECTOS-GERAIS-DO-TRATADO-A-ARTE-DE-NAVEGAR-(1606)-DE-SIMAO-D-OLIVEIRA

Panossian, M. L., Marco, F. F., Lopes, A. R. L. V., Souza, F. D., & Moretti, V. D. (2019). A atividade orientadora de ensino como pressuposto teóricometodológico de pesquisas. *Revista Reflexão e Ação*, *25*(3), 279-298. https://doi.org/10.17058/rea.v25i3.9765

Pereira, A. C. C., & Saito, F. (2019a). A reconstrução do báculo de Petrus Ramus na interface entre história e ensino de matemática. *Cocar*, *13*(25), 342-372.

Pereira, A. C. C., & Saito, F. (2019b). Os conceitos de perpendicularidade e de paralelismo mobilizados em uma atividade com o uso do báculo (1636) de Petrus Ramus. *Educação Matemática Pesquisa*, *12*(1), 405-432. https://doi.org/10.23925/1983-3156.2019v21i1p405-432

Pereira, A. C. C., & Saito, F. (2018). Os instrumentos matemáticos na interface entre história e ensino de matemática: compreendendo o cenário nacional nos últimos 10 anos. *Boletim Cearense de Educação e História da Matemática*, *5*(14), 109-122. https://doi.org/10.30938/bocehm.v5i14.225

Reis, A. E. (2003). Os instrumentos de medida. *As Novidades do Mundo: conhecimento e representação na Época Moderna* (pp. 145-167). Colibri.

Rigon, A. J., Asbabr, F. da S. F., & Moretti, V. D. (2016). Sobre o Processo de Humanização. In M. O. Moura (Org). *A atividade pedagógica na teoria histórico-cultural* (pp. 15-50). Autores associados.

Saito, F. (2009). Algumas considerações historiográficas para a história dos instrumentos e aparatos científicos: o telescópio na magia natural. In A. M. Alfonso-Goldfarb, J. L. Goldfarb, M. H. M. Ferraz, & S. Waisse, S. (Orgs.). *Centenário Simão Mathias: documentos, métodos e identidade da história da ciência* (pp. 103-120). PUCSP.

Saito, F. (2019). A reconstrução de antigos instrumentos matemáticos dirigida para formação de professores. *Educação: Teoria e Prática*, *29*(62), 571-589. https://doi.org/10.18675/1981-8106.vol29.n62.p571-589

Saito, F. (2016a). Construindo interfaces entre história e ensino da matemática. *Ensino da Matemática em Debate*, *3*(1) 3-19.

Saito, F., & Dias, M. S. (2013). Interface entre história da matemática e ensino: uma atividade desenvolvida com base num documento do século XVI. *Ciência & Educação (Bauru)*, *19*(1), 89-111. https://doi.org/10.1590/S1516-73132013000100007

Saito, F. (2016b). História e Ensino de Matemática: Construindo Interfaces. In: Salazar, J. F. & Guerra, F. U. *Investigaciones en Educación Matemática*. Lima: Fondo Editorial PUCP, 253-291.

Saito, F. (2014). Instrumentos matemáticos dos séculos XVI e XVII na articulação entre história, ensino e aprendizagem de matemática. *Revista de Matemática, Ensino e Cultura*, *9*(16), 25-47.

Saito, F. (2017). Número e grandeza: discutindo sobre a noção de medida por meio de um instrumento matemático do século XVI. *Ciência & Educação (Bauru)*, *23*(4), 917-940. https://doi.org/10.1590/1516-731320170040012

Saito, F., & Pereira, A. C. C. (2019). *A elaboração de atividades com um antigo instrumento matemático na interface entre história e ensino*. Livraria da Física.

Santos, A. G. dos. (2022). *Os aspectos matemáticos relacionados à média geométrica que emergem a partir da manipulação da escala dos números (1623) elaborada por Edmund Gunter com licenciandos em matemática* [Dissertação de Mestrado, Instituto Federação de Educação, Ciência e Tecnologia do Ceará]. Fortaleza.

Santos, A. G. dos, & Pereira, A. C. C. (2021). Descrição das escalas do Cross-Staff (1623) de Edmund Gunter. *Boletim Cearense de Educação e*

História da Matemática, *8*(23), 707-720. http://dx.doi.org/10.30938/bocehm.v8i23.4922

Silva, I. C. da, & Pereira, A. C. C. (2021). Definições e critérios para uso de textos originais na articulação entre história e ensino de matemática. *Boletim de Educação Matemática –Bolema [online]*, *35*(69), 223-241. https://doi.org/10.1590/1980-4415v35n69a11

Silva, L. P. da. (1945). A arte de navegar dos Portugueses, desde o Infante a D. João de Castro. In: *Obras completas*. Vol. II (pp. 223-432). Agência Geral das Colónias.

Silveira, Ê., & Marques, C. (2013). *Matemática: compreensão e prática. Obra em quatro volumes para os 6º, 7º, 8º e 9º anos do ensino fundamental II*. Moderna.

Van Helden, A., & Hankins, T. L. (1994). Introduction: Instruments in the History of Science. *Osiris*, *9*(1), 1-6. https://doi.org/10.1086 / 368726

Vigotski, L. S. (2001). *A formação do pensamento e da linguagem*. Martins Fontes.