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Secondary School Students' Performances on Ratio Comparison Problems

Salvador Castillo [a]
Ceneida Fernández [a]

[a] Universidad de Alicante, Departamento de Innovación y Formación Didáctica, Alicante, España

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ABSTRACT

Background: Proportional reasoning is a fundamental part of the Primary and Secondary Education curriculum. However, research has shown that both primary and secondary students find it difficult to reason proportionally. Most of these studies have used missing-value proportional problems, so studies focused on how primary and secondary school students use the concepts of ratio and proportion when solving ratio comparison problems are scarce. **Objectives**: This study characterises how secondary school students (12-16 years old) solve ratio comparison problems. **Design**: The instrument used to collect data was two ratio comparison problems with intensive quantities that could be solved as a couple of expositions or compositions. **Settings and participants**: 248 secondary education students from 7th to 10th grade (12-16 years) solved these problems. **Data collection and analysis**: An inductive analysis was carried out in order to identify categories of students' performances. **Results**: Three types of students' performances were identified depending on whether the students showed an understanding of the relative quantities: relative comparison, relative trend, and non-relative comparison. The subcategories identified in the relative trend performance showed difficulties with the critical components of the problems: difficulties with the referent and with the norming techniques. **Conclusions**: Results showed that, at the end of secondary education, students' difficulties in understanding the ratio concept and the intensive quantities persisted.

Keywords: Ratio; Intensive quantities; Ratio comparison; Secondary education.

Corresponding author: Salvador Castillo. Email: salvador.castillo@ua.es

Caracterización de respuestas de estudiantes de educación secundaria en problemas de comparación de razones

RESUMEN

Antecedentes: El razonamiento proporcional es una parte fundamental del currículo de Educación Primaria y Secundaria. Sin embargo, las investigaciones han mostrado que estudiantes, tanto de primaria como de secundaria, tienen dificultades para razonar proporcionalmente. La mayoría de estas investigaciones han usado situaciones proporcionales de valor perdido, teniendo poca información sobre cómo los estudiantes de primaria y secundaria dan significado y utilizan los conceptos de razón y proporción cuando resuelven problemas proporcionales de comparación de razones. **Objetivos**: Este estudio se centra en caracterizar cómo estudiantes de educación secundaria (12-16 años) resuelven problemas de este tipo. **Diseño**: Se diseñaron como instrumento de recogida de datos dos problemas de comparación de razones con cantidades intensivas que se podían resolver por parejas de exposiciones o composiciones. **Entorno y Participantes**: Estos problemas fueron resueltos por 248 estudiantes de Educación Secundaria de 1º hasta 4º curso (12-16 años). **Datos recopilados y análisis**: Se llevó a cabo un análisis inductivo de generación de categorías con el objetivo de identificar las distintas actuaciones de los estudiantes. **Resultados**: Se identificaron tres tipos de actuaciones en función de si los estudiantes mostraban comprensión de las cantidades relativas: comparación relativa, tendencia relativa y comparación no relativa. Las subcategorías identificadas en las actuaciones de tendencia relativa mostraron dificultades con las componentes críticas de los problemas: dificultades con el referente y con las técnicas de normalización. **Conclusiones**: Los resultados muestran que, al final de la educación secundaria, los estudiantes siguen teniendo dificultades en la comprensión de razón y de las cantidades intensivas.

Keywords: Razón; Cantidades Intensivas; Comparación de razones; Educación Secundaria.

Caracterização das respostas dos alunos do 12 ao 16 anos de idade em problemas de comparação de razões

RESUMO

Contexto: O raciocínio proporcional é uma parte fundamental do currículo do ensino fundamental e médio. No entanto, as investigações têm mostrado que os alunos tanto do ensino fundamental como do médio têm dificuldade de raciocinar proporcionalmente. A maioria dessas investigações utilizou situações proporcionais de valor omisso, mas há pouca informação disponível sobre como os alunos do ensino fundamental e médio dão sentido e utilizam os conceitos de razão e proporção na resolução de problemas proporcionais de comparação de razões. **Objetivos**: Este estudo centra-se na caracterização de como os alunos do 12 ao 16 anos de idade resolvem problemas deste tipo. **Design**: Dois problemas de comparação de razão com quantidades intensivas que poderiam ser resolvidos em pares de exposições ou composições foram projetados como um instrumento de coleta de dados. **Ambiente e participantes**: Estes problemas foram resolvidos por 248 alunos do 12 ao 16 anos. **Coleta e análise de dados**: Foi realizada uma análise indutiva de geração de categorias a fim de identificar os diferentes desempenhos dos alunos. **Resultados**: Três tipos de ações foram identificados com base se os alunos mostraram compreensão das quantidades relativas: comparação relativa, tendência relativa e comparação não-relativa. As subcategorias identificadas nas performances de tendência relativa mostraram dificuldades com os componentes críticos dos problemas: dificuldades com o referente e com as técnicas de normalização. **Conclusões**: Os resultados mostram que, no final do ensino médio, os alunos continuam a ter dificuldades na compreensão de razão e de quantidades intensivas.

Palavras-chave: Razão; Quantidades intensivas; Comparação de razões; Ensino fundamental e médio.

INTRODUCTION

The understanding of the concepts of ratio, proportion and the development of proportional reasoning have been extensively studied since the 1980s (Cramer & Post, 1993; Fernández, 2009; Gómez, 2016; Howe et al., 2011; Karplus et al., 1983; Lamon, 2012; Lesh et al., 1988; Lobato & Ellis, 2010; Noelting, 1980a, 1980b; Simon & Placa, 2012). Proportional reasoning involves comparing quantities in relative terms using ratios (Karplus et al., 1983).

Proportional reasoning is a fundamental part of the primary and secondary education curriculum. It is not only found in the different blocks of the mathematics curriculum but also appears in other areas such as physics, economics, chemistry or drawing. Research has shown that students, both at primary and secondary levels, have difficulties distinguishing proportional situations from those that are not, and the effect of some variables of problems, such as context or integer or non-integer ratios on levels of success and strategies used by students (Alatorre & Figueras, 2005; Fernández & Llinares, 2012; Fernández et al., 2012; Jiang et al., 2017; Modestou & Gagatsis, 2007; Van Dooren, De Bock, et al., 2005, Van Dooren, De Bock & Verschaffel, 2010). In fact, these difficulties are also observed in the adult population, including students training to be teachers (pre-service teachers) and in-service teachers (Buforn & Fernández, 2014; Buforn et al., 2020; Burgos, Beltrán-Pellicer, et al., 2018; Burgos, Godino & Rivas, 2019; Lamon, 2007; Rivas et al., 2012; Valverde & Castro, 2009). This means that when working on the concepts of ratio and proportion, teachers focus only on teaching their students procedural techniques to face problems involving these concepts (Burgos, Godino, & Rivas, 2019).

Most previous research has focused on showing characteristics of the development of proportional reasoning of primary and secondary students in proportional situations of missing-value (Fernández & Llinares, 2012; Modestou & Gagatsis, 2007; Van Dooren et al., 2005). This type of problems presents an equality of ratios in which three elements are known, and the fourth must be sought. However, there is little information on how primary and secondary students give meaning and use the concepts of ratio and proportion when solving proportional problems of numerical comparison or ratio comparison (Alatorre & Figueras, 2005; Fernández, 2009; Nunes et al., 2003). In these problems, two ratios whose numerical information is given in full are compared. This study will focus on these problems.

THEORETICAL FRAMEWORK

Concept of Ratio

Freudenthal (1983) defines ratio as "a function of an ordered pair of numbers or magnitude values" (p. 180). In this way, the meaning of ratio does not lie in its algorithm as a quotient by which it is assigned a numerical value, but in the comparison of ratios, being able to speak of equality or inequality of ratios without knowing the size of the ratio, so one can say that A is to B as C is to D without calculating the values A/B and C/D (Freudenthal, 1983, p. 180).

One of the challenges in its teaching is that it involves the appearance of intensive quantities that cannot be measured directly (Simon & Placa, 2012). Extensive quantities represent directly measurable attributes of an object so that extensive quantities can be added (mass, distance, volume), while intensive quantities are those that represent attributes that cannot be directly measured or added (density, price) (Piaget, 1952; Schwartz, 1988; Simon & Placa, 2012; Thompson, 1994). In general, the quotient between two extensive quantities results in an intensive one. Nunes et al. (2003) showed that primary school students find it challenging to solve proportional numerical comparison problems with intensive quantities. This study concluded that intensive quantities are more difficult than extensive quantities for two reasons: they require understanding the dependency relationships between intensive quantity and its extensive components and the ability to think in relative terms to work with intensive quantities.

In developing the understanding of ratio, Freudenthal (1983) underlines the importance of two ideas: relatively and norming. The idea of *relatively* in the sense of "put something in relation to" implies using the term ratio as a number that relates two quantities in one situation and projects this relationship into a second situation in which the relationship between the two quantities remains the same (Smith, 2002). For example, class A has 30 pupils, of which 20 are girls and 10 are boys, while class B has 60 pupils, of which 40 are girls and 20 are boys. If you ask about which class has more students, if you do not relativise, the answer will be B since the quantity is higher in absolute terms, but you really expect a comparison in relative terms where the answer would be that the number of girls in both classes is the same. On the other hand, *norming* describes the process of reconceptualising a system in relation to a fixed or standard unit. For example, imagine that the Earth is the size of a pinhead (1 mm in diameter) and then reconceptualise the solar system in terms of this definition (Freudenthal, 1983; Lamon, 1993).

Ratio Comparison Problems

Ratio comparison problems involve the ideas of *relatively* and *norming*. In these problems, there are multiplicative relationships between quantities, which express relative quantities, that is, "quantities that are in a multiplicative relationship with another quantity of reference" (which is known as "referent") (Gómez & García, 2015; p. 267). In addition, these situations involve norming techniques that make it possible to compare the intensive quantities generated by the ratios.

According to Freudenthal (1983), situations that can be relativised, such as problems of comparison of ratios, must be considered in a broader context than that of relationships within and between magnitudes, encompassing them in comparison of couples of *expositions* or *compositions*. The couple of expositions consist of a set of elements (x_1, x_2, …), with two functions ω_1 and ω_2, which assign a magnitude to each element of the set, so that the ratios $\omega_1(x_i)/\omega_2(x_i)$ are compared. An example of couple of expositions would be the comparison between runner A, who runs 100 m in 10 s, and runner B, who runs 150 m in 15 s. The runners are elements of the set Ω = {runner A, runner B}, where each is associated with two magnitudes, distance (M_1) and time (M_2) (extensive quantities), faced with functions ω_1 and ω_2, respectively. In this way, function ω_1 assigns runners A and B the 100 and 150 meters they run, respectively, while function ω_2 assigns them the 10 and 15 seconds it takes them to run these distances. The ratios to be compared (intensive quantities) are distance/time = 100/10 of runner A and distance/time = 150/15 of runner B.

The couples of compositions consist of the parts x_1 and x_2 of two universes Ω_1 and Ω_2, with two functions ω_1 and ω_2, each function representing a magnitude, so that the ratios $\omega_i(x_1)/\omega_i(x_2)$ are compared. A situation of couples of compositions would be the comparison between two concrete mixtures, Ω_1 and Ω_2, whose elements are water and cement (Ω_1 = {water, cement} and Ω_2 = {water, cement}), where the mixture Ω_1 is composed of 20dm^3 of water and 10 dm^3 of cement, and the mixture Ω_2 is composed of 30 dm^3 of water and 15 dm^3 of cement. The function ω_1 associates water and cement of the set Ω_1 their respective volumes of 20 and 10 dm^3 (extensive quantities). Likewise, the function ω_2 assigns water and cement of the set Ω_2 their respective volumes of 30 and 15 dm^3 (extensive quantities). The ratios to be compared (intensive quantities) are water volume/cement volume = 20/10 in mixture 1 and water volume/cement volume = 30/15 in mixture 2.

In these situations, norming allows unifying the antecedents (numerator) or the consequents (denominator) of the ratios to favour the comparison (Gómez, 2016). Norming can be done by employing any technique to compare ratios, such as unit rate, quotient, fraction strategy, cross product or building-up method (Cramer & Post, 1993; Hart, 1981; Lamon, 2012).

Using Sanz and Gómez's (2005) definition of the critical component, "the elements or information of the statement whose identification makes it possible to solve the problem" (p. 90), the identification of the multiplicative relationship, the identification of the equality or inequality of its ratios, and the identification of its referents are considered critical components of ratio comparison problems (Gómez & García, 2015).

A review of previous studies focusing on ratio comparison problems

The literature on ratio comparison problems has focused on showing levels of success in primary school students (Nunes et al., 2003) and on identifying student strategies and the effect of some variables of the problems, such as context or numerical structure, on the strategies of students of different educational levels (primary school students in Alatorre & Figueras, 2003; 6th to 8th-grade students in Karplus et al., 1983; 6 to 16-year-old students in Noelting, 1980a, 1980b).

Some of the correct strategies used by students in these types of problems identified in previous studies are unit rate, fraction strategy, cross product (Cramer & Post, 1993) or building-up method (Hart, 1981). Regarding incorrect strategies, it has been identified that students ignore some of the information (Ben-Chaim et al., 1998); or use inappropriate strategies in this type of problem, such as additive strategy (Cramer & Post, 1993). In addition, it has been shown that the use of one strategy or another may be conditioned by problem variables such as the presence or not of integer ratios, the size of the data, and the order in which they are presented (Monje & Gómez, 2019; Noelting, 1980a; Smith, 2002); or by the context, considering purchase contexts simpler than mixing contexts, and those that include discrete rather than continuous quantities (Tourniaire & Pulos, 1985).

However, studies focused on student performances taking into account the critical components of these problems are scarce (Gómez & García, 2015; Monje & Gómez, 2019, both studies with future teachers). Some of the performances that Gómez and García (2015) identified when future teachers solve ratio comparison problems are: obtaining and comparing ratios correctly, having difficulties with the referent, not performing the necessary operations, or comparing absolute amounts. Monje and Gómez (2019), continuing with the work of Gómez and García (2015), obtained similar performances, classifying them based on whether the future teachers made comparisons correctly, performed them incorrectly (which they called relative trend performances) or did not make comparisons.

Objective

The objective of this study is to characterise how high school students (12-16 years) solve ratio comparison problems by analysing the relationship between the critical components of these problems and students' performances.

Taking into account the research objective, we ask the following: What are the performances of secondary school students when solving ratio comparison problems?

METHOD

Participants

The participants were 248 students from a public secondary school in the province of Alicante (Spain). In particular, 68 pupils in 7th grade, 52 pupils in 8th grade, 64 pupils in 9th grade and 64 pupils in 10th grade. The number of students in each group was approximately the same. Students come from a mixed socio-economic background.

The contents on ratios and proportions in the primary education curriculum include introducing the concepts of fractions, ratios, decimal numbers and percentages, and the introduction of direct proportionality and its use in different situations. The secondary education curriculum includes the introduction of ratios and proportions, direct and inverse proportionality, the constant of proportionality, and interpreting natural numbers, integers, fractions, decimals, and percentages and operating with them through calculation strategies to use them in different situations.

Instrument

Participants solved the following ratio comparison problems:

- **Problem 1 (Offer).** In fruit store A, for every 2 kg of apples that I pay, I take 3, while in fruit store B, for every 3 kg of apples that I pay, I take 4. If the price of the kg is the same in both fruit stores, which offer is more advantageous?
- **Problem 2 (Mixture).** To make chocolate shakes, you need milk and chocolate. Juan used 450 ml of milk and obtained 600 ml of shake, while María used 750 ml of milk and obtained 900 ml. If both of them used the same grams of chocolate, which one will have a stronger chocolate taste?

Both problems are situations of inequality of ratios that can be interpreted as couple of expositions and couple of compositions; and, in both, norming techniques are necessary for comparison.

Each problem has two elements and three extensive quantities (Tables 1 and 2). In both problems, one of the extensive quantities appears unified, i.e., its value is the same for both objects. In problem 1, although not explicit, this quantity is *the amount free* (1kg in fruit stores A and B), and in problem 2, which was explicit, it is the *chocolate amount* (150ml in Juan and María's shakes).

Table 1

Extensive quantities of Problem 1.

Elements	Fruit store A	Fruit store B
Extensive quantities	Amount paid (2 kg)	Amount paid (3 kg)
	Amount free (1 kg)	Amount free (1 kg)
	Amount purchased (3 kg)	Amount purchased (4 kg)

Table 2

Extensive quantities of Problem 2.

Elements	Juan's shake	María's shake
Extensive quantities	Chocolate amount (150 ml)	Chocolate amount (150 ml)
	Milk amount (450 ml)	Milk amount (750 ml)
	Shake amount (600ml)	Shake amount (900ml)

Table 3

Couple of expositions: Amount paid (kg) / Amount purchased (kg).

	Fruit store A	Fruit store B
ω_1:Ω → Amount paid	$P_A = 2$	$P_B = 3$
ω_2:Ω → Amount purchased	$PU_A = 3$	$PU_B = 4$
Compared Ratios	$\frac{P_A}{PU_A} = \frac{2}{3}$	$\frac{P_B}{PU_B} = \frac{3}{4}$

In the first problem, if the problem is interpreted as couples of expositions, there will be a set of fruit stores Ω = {fruit store A, fruit store B}. Function ω_1 could assign to each fruit store the amount paid (P). The function ω_2 would assign to each fruit store the amount purchased (PU) (Table 3). On the other hand, function ω_1 could also assign to each fruit store the amount free (F) and function ω_2 would assign the amount purchased (Table 4).

Table 4

Couple of expositions: Amount free (kg) / Amount purchased (kg).

	Fruit store A	**Fruit store B**
$\omega_1:\Omega \rightarrow$ Amount free	$F_A = 1$	$F_B = 1$
$\omega_2:\Omega \rightarrow$ Amount purchased	$PU_A = 3$	$PU_B = 4$
Compared Ratios	$\frac{F_A}{PU_A} = \frac{1}{3}$	$\frac{F_B}{PU_B} = \frac{1}{4}$

Table 5

Couple of compositions: Amount free (kg) / Amount paid (kg).

	Amount free	**Amount paid**	**Compared Ratios**
$\omega_1:\Omega_A \rightarrow$ Quantity in kgs (Fruit store A)	$F_A = 1$	$P_A = 2$	$\frac{F_A}{P_A} = \frac{1}{2}$
$\omega_2:\Omega_B \rightarrow$ Quantity in kgs (Fruit store B)	$F_B = 1$	$P_B = 3$	$\frac{F_B}{P_B} = \frac{1}{3}$

Considering the problem as a couple of compositions, the amount purchased in both fruit stores whose elements would be amount paid and amount free (Ω_A = {amount paid, amount free} and Ω_B = {amount paid, amount free}) would be compared. The function ω_1 associates the amount free and the amount paid at Fruit store A with its respective kg and the function ω_2 associates the amount free and the amount paid at Fruit store B with its respective kg (Table 5).

The second problem, like problem 1, can be solved by couples of expositions or compositions. As couple of expositions, we have the set of shakes Ω = {Juan's shake, María's shake}. Function ω_1 assigns each shake the amount of chocolate (C) (Table 6) or the amount of milk (M) (Table 7); and function ω_2 assigns each shake the total quantity (ml) of shake (S).

Table 6

Couple of expositions: Chocolate amount (ml) / Shake amount (ml).

	Juan's shake	**Maria's shake**
ω_1: $\Omega \rightarrow$ Chocolate amount	$C_J = 150$	$C_M = 150$
ω_2: $\Omega \rightarrow$ Shake amount	$S_J = 600$	$S_M = 900$
Compared Ratios	$\frac{C_J}{S_J}=\frac{150}{600}$	$\frac{C_M}{S_M}=\frac{150}{900}$

Table 7

Couple of expositions: Milk amount (ml) / Shake amount (ml).

	Juan's shake	**Maria's shake**
ω_1:$\Omega \rightarrow$ Milk amount	$M_J = 450$	$M_M = 750$
ω_2:$\Omega \rightarrow$ Shake amount	$S_J = 600$	$S_M = 900$
Compared Ratios	$\frac{M_J}{S_J}=\frac{450}{600}$	$\frac{M_M}{S_M}=\frac{750}{900}$

If the problem is solved as a couple of compositions, each one's shake (Juan and María) whose elements are the amount of chocolate and the amount of milk (Ω_J = {chocolate amount, milk amount} and Ω_M ={chocolate amount, milk amount} would be compared. Function ω_1 associates the chocolate amount and the milk amount in Juan's shake to their respective ml, and function ω_2 associates the chocolate amount and the milk amount in María's shake to their respective ml (Table 8).

Table 8

Couple of compositions: Chocolate amount (ml) / Milk amount (ml).

	Chocolate amount	**Milk amount**	**Compared Ratios**
ω_1: $\Omega_J \rightarrow$ Volume (ml) (Juan's shake)	$C_J = 150$	$M_J = 450$	$\frac{C_J}{M_J} = \frac{150}{450}$
ω_2: $\Omega_M \rightarrow$ Volume (ml) (María's shake)	$C_M = 150$	$M_M = 750$	$\frac{C_M}{M_M} = \frac{150}{750}$

Students solved these problems individually in the usual math class schedule, in an approximate time of about 30 minutes. They were not given any instructions except that they should justify their answers and that they could use a calculator.

Analysis

An inductive category generation analysis was performed (Strauss & Corbin, 1994). Three researchers independently analysed the answers given by the students taking into account the critical components of the problems to generate categories that characterise the performances of the students:

- The idea of "relatively". Whether the student identifies the relative quantities, that is, quantities that are in a multiplicative relationship with another quantity of reference. We considered whether students:
    - Identified the multiplicative relationship.
    - Identified the referent in the comparison.

- The idea of “norming”. Whether the student uses some norming technique to compare the ratios.

The disagreements were discussed until a consensus was reached. The categories identified will be explained and exemplified in the results.

RESULTS

First, the students’ performances are described and exemplified according to the categories and subcategories identified in the analysis. Second, we show the frequencies of these categories by problem and grade.

Student Performances

From the analysis carried out, the actions can be grouped into three general categories, as in the study by Monje and Gómez (2019) with pre-service teachers, according to whether the students identify the relative quantities: relative comparison, relative trend and non-relative comparison.

- Relative comparison: This category includes student performances that show an understanding of relative quantities.
- Relative trend: This category includes student performances that showed signs of understanding relative quantities without achieving success.
- Non-relative comparison: This category includes student performances that showed no indication of an understanding of relative quantities.

The subcategories identified within each of these categories are described below. Since these subcategories were the same in both problems, they are exemplified by student answers to Problem 1.

Relative comparison

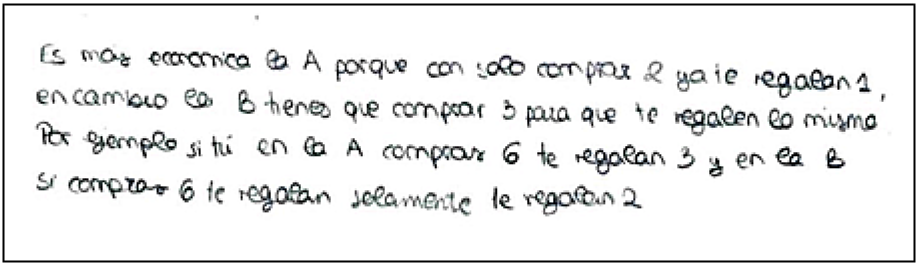
It includes the actions of students who are able to obtain the ratios to be compared after applying a norming technique. The subcategories identified are differentiated by interpreting the problem as a couple of expositions or compositions and by the referent used.

The “as a couple of compositions” subcategory captures the performances of students interpreting the problem as a couple of compositions using the ratio of *Amount free (kg) / Amount paid (kg)* in each of the offers of Problem 1 and the ratio of *Chocolate amount (ml) / Milk amount (ml)* in each shake of Problem 2.

For example, Figure 1 shows the answer of a 9th-grade student who uses as a norming technique to compare the ratios (amount free / amount paid) the search for a common multiple (6 kg that I pay), so in Fruit store A he gets 1 × 3 = 3 free kg, and in Fruit store B he gets 1 × 2 = 2 free kg.

Figure 1

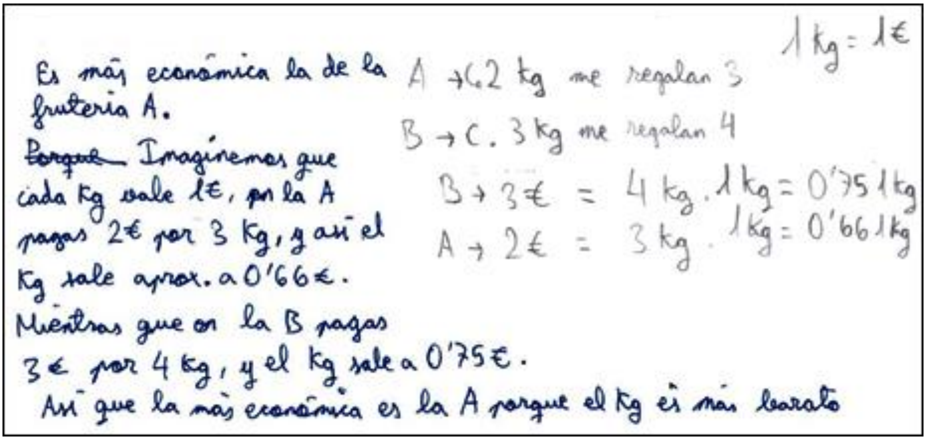
Answer from a 9th-grade student who interprets the problem as a couple of compositions.


Es mas economica la A porque con solo comprar 2 ya te regalan 1, encambio en la B tienes que comprar 3 para que te regalen lo mismo. Por ejemplo si tu en la A compras 6 te regalan 3 y en la B si compras 6 te regalan solamente te regalan 2

In the subcategory “as a couple of expositions” are the performances of students who interpret the problem as a couple of expositions. The ratios used by the students are *amount paid (kg) / amount purchased (kg)*, *amount free (kg) / amount purchased (kg)* and *amount paid (€) / amount purchased (kg)* in Problem 1, and *chocolate amount (ml) / shake amount (ml)* and *milk amount (ml) / shake amount (ml)* in Problem 2, so, in each problem, the performances differ in the referent used in the comparison. Figure 2 shows the answer of a 7th-grade student who sets a price of 1€ per kg paid, calculating the price that is paid in each fruit store with respect to the kg that he carries, obtaining the unit rate of each one (2€/3kg = 0.66€/kg in Fruit store A and 3€/4kg = 0.75€/kg in Fruit store B).

Figure 2

Answer from a 7th-grade student who interprets the problem as a couple of expositions.


Es más económica la de la frutería A.
~~Porque~~ Imaginemos que cada kg vale 1€, por la A pagas 2€ por 3 kg, y así el kg sale aprox. a 0'66€.
Mientras que en la B pagas 3€ por 4 kg, y el kg sale a 0'75€.
Así que la más económica es la A porque el kg es más barato

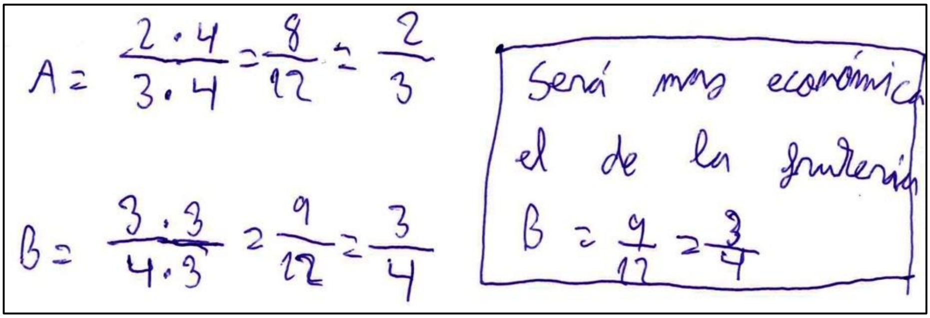
1 kg = 1€
A → C. 2 kg me regalan 3
B → C. 3 kg me regalan 4
B → 3€ = 4 kg. 1 kg = 0'75 1 kg
A → 2€ = 3 kg. 1 kg = 0'66 1 kg

Relative trend

This category includes students' performances that show evidence of understanding relative quantities, but they had problems with some critical components of the problems. Two subcategories were identified: difficulty with the referent and difficulty with the norming technique.

Figure 3

Answer from an 8th-grade student with difficulty with the referent.


$A = \frac{2 \cdot 4}{3 \cdot 4} = \frac{8}{12} = \frac{2}{3}$

$B = \frac{3 \cdot 3}{4 \cdot 3} = \frac{9}{12} = \frac{3}{4}$

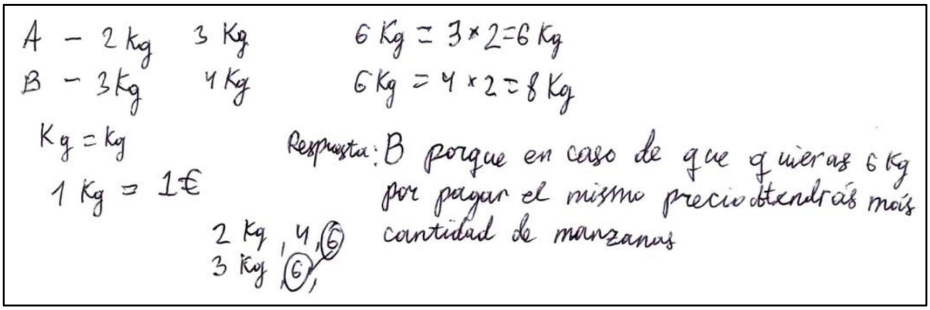
Será más económic[a] el de la frutería $B = \frac{9}{12} = \frac{3}{4}$

In the subcategory "difficulty with the referent", students can obtain the ratios to be compared by applying a norming technique, but do not correctly interpret the meaning of the ratios in relation to the referent. For example, in Figure 3, an 8th-grade student performs norming using the fraction strategy (looking for equivalent fractions with the same denominator 8/12 and 9/12), but finds it difficult to interpret the antecedent in relation to the referent (consequent – in our case, "amount purchased"), so that, although in Fruit store A he pays less (8 kg) for what he carries (12 kg) than in Fruit store B (pays 9 kg and takes 12 kg), the student answers that B is cheaper. Therefore, the difficulty is related to the loss of meaning of the referent when norming techniques are applied (Gómez & García, 2015).

In the subcategory "difficulty with norming techniques", the difficulties are related to the use of these techniques. For example, in Figure 4, a 9th-grade student uses the search for a common multiple (6 kg) as a norming technique to unify the amounts paid, but he finds it hard to extend it to the amounts purchased.

Figure 4

Answer from a 9th-grade student with difficulty in the norming technique.



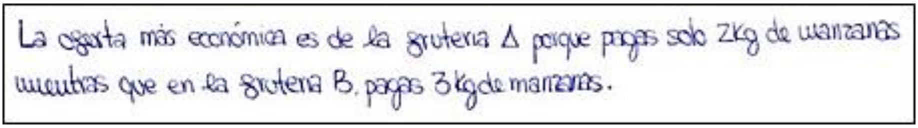
Non-relative comparison

Five subcategories were obtained: ignoring data, additive answers, affective answers, incomprehensible answers, and blank answers.

The subcategory "answers ignoring part of the data" includes answers from students who have answered by paying attention to only some of the problem data, without identifying the relative quantities. For example, Figure 5 shows an answer in which a 10th grader makes comparisons only between the amounts paid, ignoring that the offers are subject to the multiplicative relationship with the quantity of reference.

Figure 5

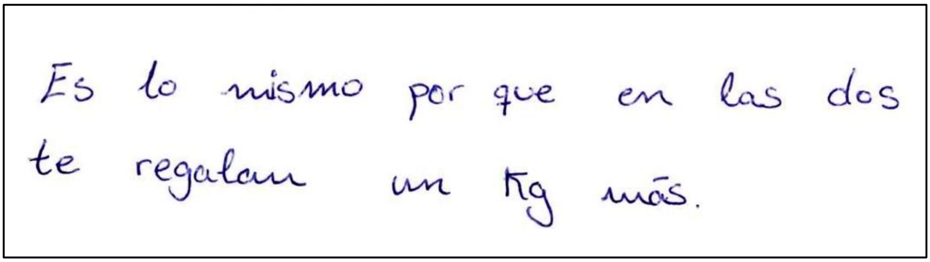
Answer from a 10th-grade student who pays attention only to the amounts paid.


La oferta más económica es de la frutería A porque pagas solo 2kg de manzanas mientras que en la frutería B. pagas 3kg de manzanas.

In the subcategory "additive answers", students relate quantities in an additive manner (they reason in absolute terms) without identifying the multiplicative relationship. Figure 6 shows the answer of an 8th-grade student who replies that both offers are the same since in both, they give one kg more than what is paid, i.e., he has subtracted the amounts paid to those purchased, obtaining the amounts free from both fruit stores, which are the same, and comparing them.

Figure 6

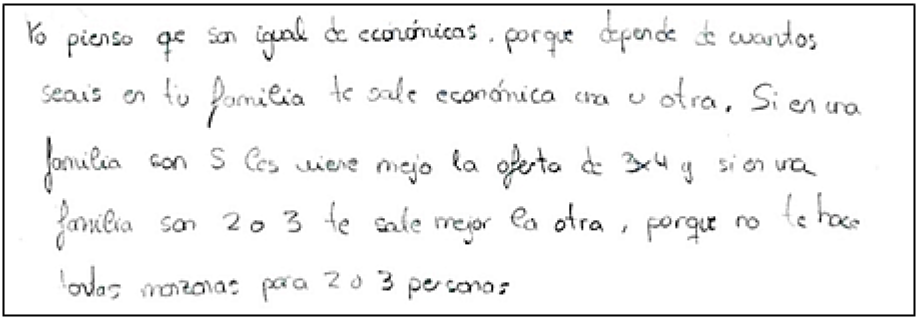
Additive answer from an 8th-grade student.


Es lo mismo por que en las dos te regalan un Kg más.

The type of answers in the subcategory "affective answers" are based on students' personal tastes or subjective interpretations. Figure 7 shows the answer of a 9th-grade student who points out that the choice will depend on the number of members the family that makes the purchase has.

Figure 7

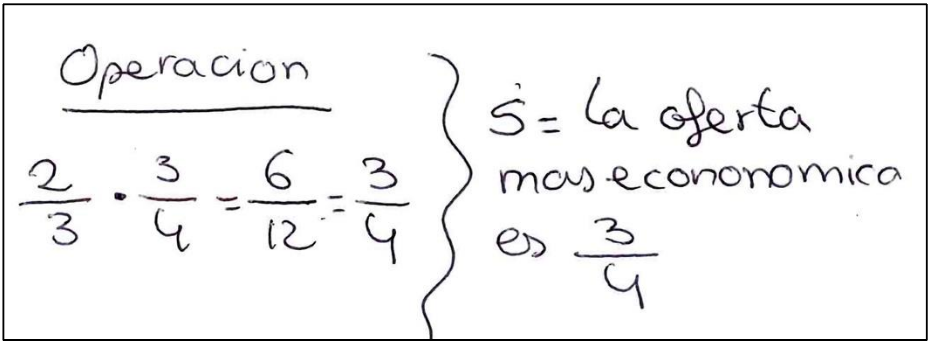
Affective answer from a 9th-grade student.



Answers in the category “incomprehensible answers” are those answers in which students perform meaningless operations. An example of this type of answer is shown in Figure 8. An 8th-grade student obtains a fraction for each offer but then multiplies them illogically.

Figure 8

Incomprehensible answer from a 8th-grade student.



Students’ actions by problem and grade

Table 9 shows the total percentage of relative comparison, relative trend, and non-relative answers in each of the problems by grade.

Table 9

Percentage of each of the answer categories by problem and grade.

Grade	Problem 1 (Offers) Relative comparison	Relative trend	Non-relative	Problem 2 (Mixtures) Relative comparison	Relative trend	Non-relative	Total
7th	25.0%	14.7%	60.3%	70.6%	7.4%	22.0%	47.8%
8th	25.0%	19.2%	55.8%	86.5	3.8%	9.7%	55.8%
9th	20.3%	18.8%	60.9%	82.8%	4.7%	12.5%	51.6%
10th	50.0%	15.6%	34.4%	84.4%	1.6%	14.0%	67.2%
Total	30.1%	17.0%	52.9%	81.1%	4.4%	14.5%	55.6%

Table 10

Percentage of subcategories identified for Problems 1 and 2.

Categories and Subcategories	Problems 1	2
Relative comparison		
As a couple of compositions	6.8%	68.2%
As a couple of expositions	23.3%	12.9%
Relative trend		
Difficulty with referent	5.6%	2.0%
Difficulty with norming	11.4%	2,4%
Non-relative comparison		
Answer in which they ignore data	19.0%	4.3%
Additive answer (absolute)	8.1%	4.3%
Affective answer	7.3%	1.6%
Incomprehensible or blank answer	18.5%	4.3%

Although there is an increase in relative comparison answers over the years, success is 67.2% (average for both problems) in the 10^{th} grade, which shows that students' difficulties in the problems comparing ratios (or with intensive quantities) do not go away throughout secondary education.

Table 10 shows the percentage of performance subcategories identified for Problems 1 and 2.

In Problem 1 (offers), 30.1% of the answers were classified as relative comparison. Only 6.8% interpreted the problem as a couple of compositions. The rest, 23.3%, interpreted the problem as a couple of expositions. The students' difficulties in this problem are shown by the 17% who employed answers of relative trend, having difficulties with the referents and with the norming techniques, and by the 52.9% who provided non-relative answers. Of the latter group, the emergence of answers that ignored some data or provided incomprehensible or blank answers stands out.

In Problem 2 (mixtures), 81.1% of the answers were classified as relative comparison. In particular, 68.2% interpreted the problem as a couple of compositions and 12.9% as a couple of expositions. 4.4% were relative trend answers presenting the same types of difficulties as in the first problem. Finally, 14.5% were non-relative comparisons, whose most frequent subcategories were ignoring problem data, incomprehensible or blank answers, and additive answers.

CONCLUSIONS AND DISCUSSION

The results obtained provide information on the performances of secondary school students when they face proportional problems of numerical comparison with intensive quantities, depending on the critical components of the problems. In this way, three types of actions have been identified based on whether secondary school students identify the multiplicative relationship and, therefore, show an understanding of the relative quantities: relative comparison, relative trend, and non-relative comparison. These performances coincide with the results obtained by Monje and Gómez (2019) with pre-service teachers, where they used realistic tasks of advertising leaflet offers, and extended them to secondary school students. In addition, the subcategories identified in the relative trend performances show difficulties of students with some critical components: difficulties in interpreting the antecedent in relation to the consequent (referent) in the comparison, and difficulties with norming techniques.

On the other hand, the subcategories identified in the non-relative comparison performances show that secondary school students have difficulties identifying the multiplicative relationship, reasoning in additive (absolute) terms or ignoring part of the data. Both subcategories have been identified in previous studies (Alatorre & Figueras, 2005; Ben-Chaim et al., 1998; Fernández & Llinares, 2012; Hart, 1981; Monje & Gómez, 2019; Tourniaire & Pulos, 1985; Van Dooren et al., 2005). Regarding the additive strategy, studies have shown that they are more used in problems where non-integer ratios appear, which coincides with the ratios of the problems of this research, although some of them could have been considered integer ratios if the reciprocal ones had been built. However, it has not been used much in this research compared to studies where problems of lost value have been used (Fernández & Llinares, 2012). As for the subcategory "ignoring data", students have paid attention to only one of the given quantities, relating quantities within magnitudes in couples of expositions or between magnitudes in couples of compositions. Ben-Chaim et al. (1998) pointed out that the fact that students focus only on one of the quantities of the ratios (antecedent or consequent) when these are compared happens due to the challenge posed by proportional reasoning, since it requires "a comparison of two numbers as a single entity and operate simultaneously with two or more comparisons" (p. 262).

The characterisation of student performances obtained in this study could constitute a tool that helps to understand the students' mathematical thinking and, therefore, provides information for the design of classroom proposals that could help students overcome difficulties. Our results also have implications for the training of secondary school teachers. Burgos, Beltrán-Pellicer et al. (2018), in a proportionality study with students of a secondary teachers' master's degree, recognised that even those coming from the degree of mathematics have deficiencies in the knowledge of the content of proportionality, confusing the meaning of ratio or mistranslating its terms. This could imply that prospective secondary school teachers have difficulty interpreting their students' answers when working on proportional reasoning, as previous studies with prospective primary school teachers have pointed out (Buforn et al., 2020; Burgos, Godino & Rivas, 2019). Therefore, it is important that secondary school math teachers know the different types of performances of secondary school students when facing ratio comparison problems, since it will allow them to identify the difficulties that their students encounter and, in this way, design tasks that help them to progress in their learning.

Results on the percentages of each performance over the years indicate that although there is an evolution in the level of success, students at the end of secondary education cannot understand well relative numbers, as 32.8% (average of the two problems) of 10^{th}-grade students provided relative trend or non-relative comparison answers. Therefore, the difficulties associated with intensive quantities seem to persist during secondary education. Howe et al. (2010) and Nunes et al. (2003) showed that primary school students have difficulties comparing the ratios. Our results extend these studies showing that, although there seems to be a positive evolution throughout the secondary education grades, the difficulties persist in the final years.

In addition, the results show that students provided a greater number of relative comparison answers in Problem 2 (mixtures) than in Problem 1 (offers). This contradicts some previous research that states that mixture problems present greater difficulty (Alatorre & Figueras, 2005; Fernández, 2009; Tourniaire & Pulos, 1985), while other research has not found differences in students' actions (in this case, primary students) when facing problems of both types (Nunes et al., 2003). This result could be interpreted from the characteristics of the problems. In both problems, one of the quantities was provided to them in a unified form. In problem 1, although it was not explicit, the amount free was the same for both fruit stores (1 kg). In Problem 2, it was made explicit in the statement that the chocolate amount was the same in both shakes. Providing the quantities explicitly unified in the statement seems to allow students to respond without performing calculations, which could have facilitated success on Problem 2 by explicitly stating this amount. Future research could focus on examining whether or not students' level of success varies by explicitly asking for unified quantities, in addition to whether or not it varies if they are not unified.

On the other hand, coinciding with previous studies (Alatorre & Figueras, 2005; Fernández, 2009), Problem 1 (offers) has been solved, preferably, by couples of expositions, while Problem 2 (mixtures) has been solved by couples of compositions. In mixing problems, the two quantities provided are the parts of a whole (milk and chocolate are the parts of the shake), which could condition it to be solved by pairs of compositions. However, the interpretation of the problem of the offers should be that the amount purchased was formed by the amount paid and the amount free to be considered a pair of compositions. On the other hand, it is also possible that the way in which the quantities have been presented in the problems has influenced. In the problem of offers, the quantities explicitly presented are amount paid and amount purchased, which may have conditioned the students to form amount paid/amount purchased ratios, corresponding to couples of expositions. In the second problem, the explicit quantities are milk amount and shake amount, but when asking about chocolate flavour, students had to pay attention to the chocolate amount. Future papers could examine whether these variables influence the different actions of students.

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AUTHORSHIP CONTRIBUTION STATEMENT

Both authors, SC and CF, have contributed to the design and development of this study and to the writing of this manuscript.

DATA AVAILABILITY DECLARATION

The data supporting the results of this study are found in the authors' files. The data will be made available by the first author, SC, in charge of its custody, upon reasonable request.

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