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Spatial Thinking and Geometric Systems: Analysing Cognitive Demand in School Mathematics Tasks

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ABSTRACT

Background: The research context is the synchronous teaching of mathematics visualised through signs, as is the Colombian sign language, which is publicly accessible, especially for the schooled deaf. **Objectives**: To analyse the cognitive demand of four school mathematical tasks developed as synchronous geometry lessons proposed for the deaf schooled population in Colombia. **Design**: Qualitative research within the interpretative paradigm, in which the teaching phenomenon of synchronous geometry classes is interpreted from the content analysis of the cognitive demand from the theoretical model of school mathematics task on the classes. **Setting and Participants**: The research is developed under the non-participant observation of the video lessons hosted on the public channel of the National Institute of the Deaf in Colombia. The classes are taught by a single teacher, in which one hundred people participate, who followed the teacher's signs. **Data collection and analysis:** Of the twenty-nine synchronous lessons hosted on the channel, we selected the ones that focused on the teaching of two-dimensional geometry (triangles, their characteristics and their properties, similarity, congruence, and Pythagoras theorems) in a way that allows analysing the scholastic meanings of concepts, definitions, and procedures from the cognitive demand on these. **Results**: The results point to the development of thirty tasks centred on memorisation and using procedures without connections. **Conclusions**: The need to reflect on the lesson plans, especially for the deaf population, is evidenced, besides the search for tasks that motivate the use of procedures with connections and the construction of mathematics. Moreover, it is necessary to include the exploratory-investigative in geometry teaching.

Keywords: Spatial thinking; School mathematical tasks; Cognitive demand; Two-dimensional geometry; Triangles.

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Pensamento Espacial e Sistemas Geométricos: Análise da Exigência Cognitiva em Tarefas de Matemática Escolar

RESUMO

Contexto: O contexto da pesquisa é o ensino síncrono de uma matemática visualizada por meio de sinais, como é a língua de sinais colombianas, as quais são de acesso público, sobretudo para os surdos escolarizados. **Objetivos:** analisar a exigência cognitiva de 4 Tarefas Matemáticas escolares desenvolvidas em aulas síncronas de geometria propostas para população surda escolarizada da Colômbia. **Design**: A pesquisa qualitativa, dentro do paradigma interpretativo, na qual se interpreta o fenômeno de ensino das aulas síncronas de geometria, a partir da análise da exigência cognitiva do modelo teórico das Tarefas Matemáticas escolares nas aulas. **Ambiente e participantes**: a pesquisa é desenvolvida sob a observação não participante das videoaulas hospedadas no canal público do Instituto Nacional de surdos na Colômbia. As aulas são ministradas por um único professor, na qual participam 100 pessoas, que seguiam os sinais do professor. **Coleta e análise de dados**: Das 29 aulas síncronas e hospedadas no canal, foram selecionadas as que focavam no ensino da geometria bidimensional (triângulos, suas características e suas propriedades, teoremas de semelhança, congruência e Pitágoras) de forma que permita analisar os significados escolares dos conceitos, definições e procedimentos a partir da exigência cognitiva sobre estes. **Resultados**: Os resultados apontam para o desenvolvimento de 30 tarefas, centradas em memorização e uso de procedimentos sem conexões **Conclusões**: Se evidencia a necessidade de refletir sobre os planos de aula, sobretudo para a população surda, além da procura de tarefas que motivem o uso de procedimentos com conexões e construção das matemáticas. Além disso, é necessário incluir o exploratório-investigativo no ensino da geometria.

Palavras-chave: Pensamento espacial; Tarefas Matemáticas Escolares; Exigência cognitiva; geometria bidimensional; triângulos.

INTRODUCTION

The mathematics curriculum in Colombia is delimited by the curriculum guidelines of the Ministry of National Education (MEN, 1998) and other documents, such as the basic standards of competencies (MEN, 2006). It also comprises the students' fundamental learning rights (MEN, 2016), which promote a globalised vision of mathematics as a required area of knowledge within the institutional curriculum inside elementary and secondary schools. Those documents emphasise that the teaching of mathematics must be guided by its components called 'mathematical thoughts', which are subdivided into areas of specialisation: numerical, spatial, metric, variational, and random. In turn, these are developed from the mathematical processes of reasoning, exercising, modelling, problem solving, and communication and mediated by contexts in which learning situations can be proposed, such as the mathematical context, the interdisciplinary context -through other sciences- and the everyday context.

The general orientations motivate the formation of mathematically competent citizens, which implies that students can *know how to do* and/or apply different thoughts and processes to develop different situations (MEN, 2006). However, teachers have adapted these guidelines to the institutional autonomy governed by the area plans of each grade that make up schooling in Colombia. Therefore, it is not easy to find a single model of national education, as it will also depend on the realities of the classrooms, and the movements (migratory student and teacher population), family and social dynamics, among others that occur within the different Colombian cities, which influences and transcends the simple understanding of the mathematical objects with which we are going to work. Likewise, not all mathematical thoughts are taught with the same intensity in regular classrooms, which has projected an implicit importance of the numerical and variational over the geometric and random.

Despite the recognition of the contributions of the development of visualisation skills, critical thinking, intuition, problem solving, and argumentation that can be developed from spatial thinking, we perceive that less time is invested in those aspects compared to teaching numerical thinking. Other aspects that influence this problem are the segmented and isolated teaching of the different thoughts when they are treated as non-sequential contents of the curriculum, corresponding to difficulties inherited from a scarce of transversality teacher education or experiences located in exploratory environments (Acevedo-Rincón, 2018; González & Díaz, 2018), as evidenced in the results of national and international standardised tests carried out in Colombia, where evaluations are presented with problem-solving scenarios that require the combination of this class of thoughts, as revealed by the current research reports (Castro-Ávila & Ruiz Linares, 2019; Marmolejo-Avenía, G.A., Tarapuez-Guaítarílla, 2019; Rodríguez & Nates, 2021; Téllez, 2021; Martínez, 2021).

Not far from this problem is the education of the deaf population, not only in Colombia but in Latin America, in which they denote the limitations in implementing didactic, pedagogical, and disciplinary strategies for the population with disabilities in regular classrooms. There is also a need for specialised professionals to assist school mathematics teaching to guarantee the right to include the deaf issue, a fundamental right in Colombian citizenship, in the training processes (Ladd, 2003; Meresman & Ullmann, 2020; Muñoz Vilugrón et al., 2020; Acevedo-Rincón & Flórez-Pabón, 2022).

In this sense, usually, spoken, and written language present substantial differences that lead the hearing to find difficulties in interpreting texts and solving problems, accentuated when teachers do not have training in sign language (LSC, for Colombia). These difficulties could be overcome if teachers used visual aids (Ignatius, Nogueira & Da Silva, 2018). However, it should be clarified that the use of graphic resources by themselves does not guarantee students' understanding or the resolution of situations.

The Universal Declaration of Linguistic Rights (UDL) for the deaf population (UNESCO, 1996), which guarantees their right to participation, emerges from the needs presented to address the diversity of minority groups that have been invisible for decades. Likewise, the National Institute of the Deaf (INSOR) was established in 1997 as a public entity of national order, linked to the MEN, whose objective is to promote the development and implementation of public policies for the social inclusion of the deaf population from the public and private education sector (INSOR, s/f). These affirmative actions toward inclusion have managed to make the deaf population, and, in general, with disabilities, visible regarding their needs and the responsibilities of the territorial entities for their fulfilment (for example, c.f. decree 1421 of 2017 of the MEN, 2017).

Given this perspective, the commitment of INSOR is to maintain inclusive contact with the school population of Colombia in a hearing disability condition, so through classes transmitted live, it was possible to reinforce some content from all areas between 2017 and 2021, transmitting between five and six lessons per year in of mathematics. But, the fact of transmitting live content does not guarantee the development of the necessary mathematical skills and competencies, nor the motivation for the different reasoning of the participants, even in transmissions in which not all the participants belong to the deaf population, but are hearing people interested in accompanying the classes (parents, brothers and/or teachers of participating students) or motivated by research on the pedagogical and didactic models of teaching or learning mathematics (Riaño & Nicol, 2020; Ortega, 2020).

In accordance with these concerns, the research aims to analyse the cognitive requirement of four mathematical tasks (MT) developed during the synchronous geometry sessions for the deaf population of Colombia. So the research is oriented by the question: What are the levels of cognitive demand of MTs implemented for the teaching of geometry and the development of spatial thinking during the live classes of the INSOR channel in Colombia? Next, there is theoretical support against the levels of cognitive demand (low and high), which can be developed from the selection and implementation of MTs. Subsequently, the context of the research is presented, from which the MTs analysed according to their cognitive demand were selected. Finally, we build the conclusion around the importance of teaching geometry and spatial thinking to the deaf population in Colombia.

THEORETICAL FRAMEWORK

This research presents as a theoretical framework the bases of MTs from the cognitive demand and learning limitations in the teaching-learning process in the context of Colombian educational institutions.

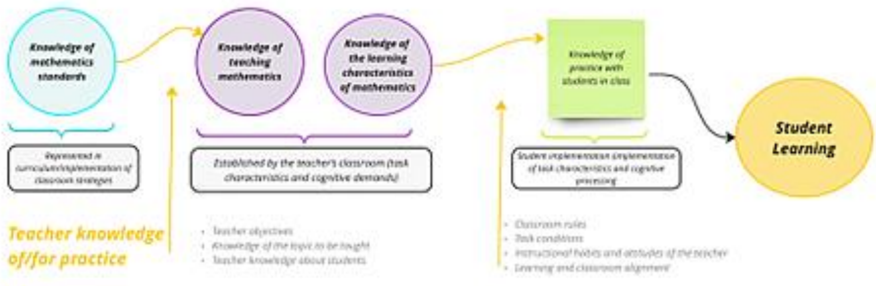
According to Stein and Smith (1998), a MT is defined as "a segment of class activity (of mathematics) that focuses on the development of a particular mathematical idea" (p. 268). However, this is not enough by itself to guarantee the learning of a specific mathematical reality, because other factors, such as objectives and class management, influence how a teacher invites students to participate in the task, which is thought from the exercise of class planning (Acevedo-Rincón, 2017).

Several factors influence the learning analysis of MTs; some are related to the teacher's practice (*knowledge of/for practice*), and others are related to the student, developed from the teacher's orientations in practice within the classroom (*knowledge in practice*), as shown in Figure 1.

This implies that the MT is conceived from the teacher's standpoint, which allows the development of the students' competencies and knowledge about specific situations. That is, according to Figure 1, the teacher's knowledge can be staged from the curricular and conceptual knowledge of mathematics and for school practice (*Knowledge of Learning Mathematics Standards* - KLMS), which allow organising a teaching planning from the knowledge of mathematics didactics, but also how students learn. At this point, the school math task manifests itself in a state of organisation (planning) of the characteristics of the tasks and the cognitive demands that involve such planning. Finally, the above results in a knowledge in practice that the teacher implements when guiding and investigating the task and possible spectra of responses of the school math task, which allow students to identify issues specific to the task, and reason on the answers until consolidating their learning. These aspects can enrich the development of MTs based on class rules, teacher instructions, attitudes, personal relationships between the teacher and student, the learning rhythms of the students and their commitment to the class.

Figure 1

Relationship Between the Variables Incorporated in the Task and Students' Learning. (Adapted from Contributions to the initial scheme of Stein, Grover, & Henningsen, 1996, p. 459)



Other perspectives that enrich this vision of MTs correspond to what was proposed by Moreno and Ramírez-Uclés (2016), in which the task is a mathematical action of the student, previously planned by the teacher as a learning or evaluation element, or the proposal by Ramos-Rodríguez, Valenzuela-Molina and Flores (2019). in which its development is linked to the relationships of the student-teacher-content triad.

According to Smith and Stein (1998), the analysis of MTs can be carried out from primary descriptors, such as what is observed at first sight of MTs and secondary descriptors, which imply an in-depth analysis of the tasks, their purpose, and mathematical content necessary for their development. It also implies studying the coherence between task instruction and its purpose to investigate learning limitations (errors and difficulties) and cognitive demand (Smith & Stein, 1998). This presents the taxonomy of cognitive demand generated by MTs, which are divided into low-level requirements and recognise memorisation and procedures without connections as the basis for the development of a task, and high-level ones, which recognise procedures with connections and the construction of mathematics as elements involved in the development of MTs.

Figure 2

Taxonomy of the Cognitive Demand of MT*s*. (Adapted from Smith and Stein, 1998, p. 348)

As highlighted in Figure 2, cognitive demand for the task is based on the characteristics of previous outcomes (Doyle, 1988; Resnick, 1987; NCTM, 1991; Stein, Grover & Henningsen, 1996; Stein, Lane & Silver, 1996). These constitute an adequate conceptual basis for the analysis of MTs, in which the processes of mathematical competency development are limited to the planning and offer of MTs that motivate the development of high-level cognitive requirements in which the procedures and learning of concepts transcend the simple exercise of algorithms or the memorisation of concepts and rules in the teaching-learning process.

From the descriptors of the cognitive demand of the tasks (Figure 2), it will be possible to characterise the school mathematical tasks implemented for developing mathematical thinking in the classroom. However, this is not exclusive to a school curriculum, since the video classes of INSOR are proposed in an open way, not oriented by standardisation of knowledge, but by recognising the mathematical concept from levels drawn from its planning. So, students attending the online broadcast of the video lessons can participate in the tasks within the stipulated times within the INSOR live development via live chat or WhatsApp. If these responses arrive during the broadcast, they can be fed back. Subsequently, these conversations are saved within the playlist of the mathematics classes in the institutional channel, which the students can develop in their own time without feedback from the synchronous class. However, it can be used as a reinforcement resource at home or by teachers with deaf or low-hearing students inside the schools.

On the other hand, authors such as Gutiérrez and Jaime (2013) and Gutiérrez, Jaime, and Alba (2014) have referred to the study of the characteristics of the tasks in regular classrooms under the need to assess the cognitive efforts of the students, and reciprocally involving the planning teaching practices, aligned with this objective. Therefore, the MT approach also involves thinking about the learning limitations of school geometry. Among the most well-known limitations is the lack of training of mathematics teachers with the specificity of assisting the deaf population (Barham & Bishop, 1991), the simplification of statements to transform them into more understandable formats, or the accompaniment of an interpreter who understands the concepts, signs, and translations of meanings typical of the mathematical work (Rosich & Serrano, 1998).

Balacheff (2000) affirms that the conceptions of the hearing impaired are weaker than those of hearing students in terms of linguistic implications and the classification of non-dichotomous situations and tasks involving images in the contents. The same happens with those referred to by Rosich et al. (2006), in which specific difficulties are evidenced when establishing relationships between words and images, interpreting mathematical properties, and recognising the possibilities of defining characterising geometric concepts through different properties, among others.

In addition to this, there are specific mathematical obstacles that they share with the hearing population. According to Socas (2008), they can come from different sources, such as the complexity of the mathematical object, the teaching, cognitive, or affective processes of the students compared to the learning of mathematics. Likewise, Montes, Climent, and Contreras (2022) consider that a reduced set of examples can constitute an obstacle for the student not to reach the generalisation of these concepts. In addition, different studies reveal a teaching style that, although promoting tasks for deaf people, shows a gap with those of the hearing, since the level is lower than proposed. For the latter, consider only basic cognitive processes, which do not meet the needs of the deaf (Gallo, 2011).

METHODOLOGY

This research with qualitative methodology aims to analyse the phenomenon of teaching school mathematics during the execution of MTs in the synchronous classes of the open class model for the deaf and low-hearing population in schools in Colombia. However, people participated in their development, such as parents, teachers, and family members, among others, some hearing people with knowledge of Sign Language, and others interested in knowing the structure of the class. In addition, this research is considered exploratory and descriptive (Hernández, Fernández & Baptista, 2014), as we interpret and identify the levels of cognitive demand of the tasks from each of the free access videos hosted on the institutional YouTube channel of the National Institute of the Deaf of Colombia (INSOR).

This research is carried out within the teaching framework through open classes for the Colombian deaf population. On average, six to seven classes are held annually, distributed throughout the year between 2017 and 2021. Of the 29 classes held in that period, only seven were held in 2019 and one in 2021.

Below, Table 1 presents the names and contents of the video classes, the date on which they were developed, the duration and the amount of the MTs proposed for developing class content.

Table 1

Video List of Spatial Thinking and Geometric Systems Classes. (Constructed from INSOR educational channel playlist information, https://www.youtube.com/playlist?list=PLyihWF0yVBhQklMXkBFmH2gMm8xtAjeVX)

Class Name	Date	Duration	Number of MTs per Class
Triangular Properties in LSC_Math Live Classes	10/04/2019	1h09m32s	5
Classification of Triangles in LSC_Math Live Classes	07/05/2019	1h02m29s	4
Building Triangles in LSC_Math Live Classes	06/06/2019	1h05m24s	4
Pythagorean Theorem in LSC_ Math Live Classes	09/08/2019	1h21m19s	6
Similarity of Triangles_Math Live Classes	03/09/2019	1h21m31s	5
Triangle Congruence in LSC_Math Live Classes	03/10/2019	55m18s	6

Table 1 presents each of the topics and concepts developed in 2019. In the specific case of the proposal for the development of spatial thinking based on geometry tasks, in the video classes of INSOR, six live classes were proposed, focused on the teaching of geometric concepts that are in some way general and transversal at all levels of elementary and secondary education, by the same characteristics of the population participating in the classes. From these topics, four MTs were selected for analysis involving: a. the construction of triangles given the characteristics of sides measurements, b. internal angles, c. Pythagorean theorem and d. congruence between triangles. In the case of the spatial problem-solving class, it was oriented towards identifying the views and positions of three-dimensional shapes in contrast to the approach given to two-dimensional geometry in 2019.

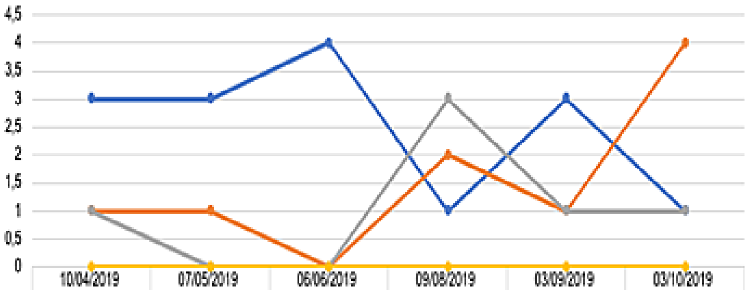
For the analysis of the video classes, we used the observation technique, which involves “paying attention to details, possessing skills to decipher and understand behaviours, being reflective and flexible to change the centre of attention" (Hernández, Fernández, & Baptista, 2014, p. 403), even more, if the language of the class is not native to the research team. This motivated an immersion of one of the researchers in courses of Colombian sign language (LSC) to understand in depth what was exposed, before an eventual silence of the audio description of the interpreter that translated the class synchronously. In addition, not all the videos are subtitled (due to the silences), which prevents a fluid data analysis. We had to repeat the same video fragment on several occasions, to understand what was happening, as Díaz-Cintas (2010) recounts regarding the need for subtitling and audio describing the video, mainly for understanding what was communicated by the teacher and/or interpreter (for the hearing). These actions also help the non-native population to put themselves in the deaf students' and teachers' shoes. According to Krippendorff (1990), with this data set, content analysis can be carried out, establishing the mathematical content corresponding to the spatial thought and the geometric systems exposed by the teacher and interpreter as a unit of analysis. Thus, this analysis is theoretical-qualitative, from the understanding of the levels of cognitive demand of four tasks proposed for the teaching of geometry and its limitations for learning (errors, difficulties, and obstacles), which will allow studying the diversity of school meanings of the concepts and their procedures (Rico & Fernández-Cano, 2013). In this sense, the structure and formal analysis of concepts, definitions, procedures, and representation systems are studied, which includes the different notations (pictorial, graphic, symbolic, and signs involved), in addition to the phenomenological analysis on teaching in virtual media for the population with disabilities during the pandemic, which, in this context, gives (totally) meaning to the mathematical contents that are the object of study (Rico & Fernández-Cano, 2013), focused on the look on the tasks proposed for the sequence of classes.

RESULTS AND ANALYSIS

For the purposes of the analysis of the MTs developed in the channel, we will focus on those that were developed to introduce, teach, or exercise a specific topic, since it allows us to evidence the order and sequence of their development during the sequence of the classes (Figure 3), and not a random treatment, such as some of the video classes of 2020 and 2021, whose focus is the resolution of problems of the different thoughts of mathematics, without generating sequentiality in their development.

Figure 3

Cognitive Demand in Video Classes of Spatial Thinking Tasks. (Construction based on INSOR educational channel playlist information, https://www.youtube.com/playlist?list=PLyihWF0yVBhQklMXkBFmH2gMm8xtAjeVX)



	10/04/2019	07/05/2019	06/06/2019	09/08/2019	03/09/2019	03/10/2019
	1	2	3	4	5	6
Bajo nivel de demanda (MEM)	3	3	4	1	3	1
Bajo nivel de demanda (PWoC)	1	1	0	2	1	4
Alto nivel de demanda (PWC)	1	0	0	3	1	1
Alto nivel de demanda (DM)	0	0	0	0	0	0

Bajo nivel de demanda (MEM) Bajo nivel de demanda (PWoC) Alto nivel de demanda (PWC) Alto nivel de demanda (DM)

In this sense, 30 MTs were proposed in the development of the geometry and spatial thinking module, which were transmitted during the year 2019. In these MTs, the low-demand tasks in blue and orange, characterised as Memorisation (MEM) and the corresponding Procedures without connections (PwoC), respectively, can be highlighted. In grey and yellow, the tasks of high cognitive demand developed in the geometry module are registered, characterised as Procedures with Connections (PWC) and Mathematics Construction (DM). Of this, we should note that, during the six sessions held in 2019, none of them was characterised as DM; only six were found in the PWC category. In contrast, the low-demand, blue and orange tasks in the graph stand out during classroom development. These are the ones that predominate in the set of tasks (24/30).

In the same sense, four related to MTs of different levels of cognitive demand are described and interpreted, as evidenced in Table 2. There, the geometric contents selected for the analysis are presented based on the approach of the levels of cognitive demand of MTs for the spatial and geometric thinking module.

Table 2

Declared Content Classification for Cognitive Demand Analysis.

MT	Purpose	Geometrical Content
1	Check the measure of the sum of the internal angles of any triangle	Internal angles.
2	Identify that the sum of two sides of a triangle must be bigger than its third side, to form a closed flat figure.	Triangular inequality.
3	Check the Pythagorean theorem from different representations.	Pythagorean theorem
4	Identify the congruence of two triangles by applying the ALA criterion.	Congruence criteria.

Note. The activities presented here were selected from the 30 developed in the module of spatial thinking and geometric systems, ensuring that they are distributed at the different levels of Cognitive Demand. There were no DM-type activities in the development of the module.

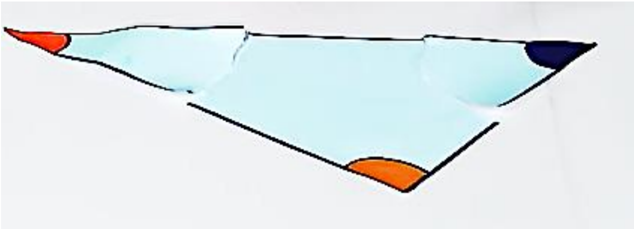
Table 2 presents the selected tasks according to the different levels of cognitive demand. Below, the five s and their respective class analyses are presented in detail.

MT 1. Figure 4 contains the representations that arise when verifying that the measurement of the internal angles is 180°. In a blue triangle, the three angles of its interior have been demarcated. The objective is to check that the sum of the angles adjacent to the orange angle is 180°. For this fact, we used the concrete representation of the abstract form (triangle).

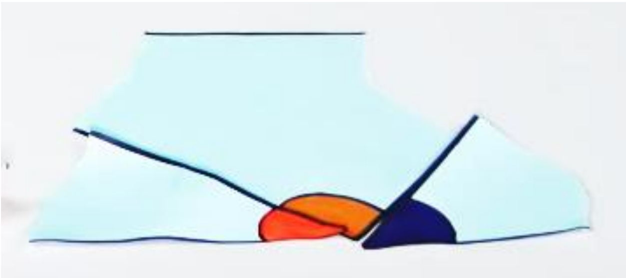
Before the class, we characterised the triangular shape by the number of sides and angles. On this occasion, we intended to generalise the value of the sum of the internal angles, whose value is constant (180º) independent of the internal angles that make up the triangle.

Figure 4

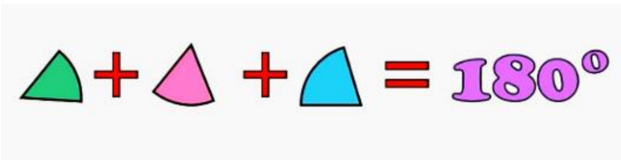
Checking the Internal Angles of the Triangle. (Adapted from class ‘triangular properties in Colombian Sign Language live class of mathematics’ of the educational channel INSOR, https://youtu.be/DEaw2yKL10c)



a.



b.



c.

For verification, five triangles (isosceles, rectangle, acute triangle, acute-angled triangle, equilateral, and obtuse-angled triangle,) are used, which lead the student to think that regardless of the characteristics of the angles or the sides of the triangles, their sum will always be 180°. For this, students must identify the highlighted angle in red, orange, or blue (Figure 4a), which they will arrange adjacent to each other (Figure 4b).

When analysing the expected learning limitations for this task, two can be identified: the incorrect associations or rigidity of thought and those that correspond to deficient learning of skills, facts, or concepts or the application of irrelevant rules or strategies.

By identifying the angle, as a space highlighted inside the triangle, the meaning of the angle is diluted as the representation of the opening between two segments and leads the student to understand it as an underlined surface, which agrees with Cañadas (2002), who pointed out the particular cases as a basis for reaching general conclusions and reinforces what was proposed by Balacheff (2000) and Rosich (2006) before the difficulty represented by the solution of tasks that involve images in the contents. But, when cutting the 'angles', possibly new triangles are formed, and not necessarily a 'representation of the angle', and if the angle has not been previously highlighted, they will hardly construct the flat angle, bringing together the three cut parts, which, in this case, would lead students to make incorrect associations or show rigidity of thought (Radatz, 1979). However, in the deaf population, this type of representation will involve conflicts when establishing relationships between images, concepts, and properties, as announced by Rosich (1998; 2006).

In the same way, the procedure is repeated with different types of triangles, which does not lead the student to inquire about the generalisation of the concept of the internal measure of the sum of the angles. Because when presenting triangles of different sizes, it is not intentionally indicated that the new triangle corresponds to a triangle with new characteristics of sides and angles, only its difference is highlighted, but it is not indicated in what lies such difference. Thus, the participant repeatedly engages in poor learning of skills, facts, concepts, or the application of irrelevant rules or strategies (Radatz, 1980). Likewise, it reveals the need for memorisation of procedures, with no connection close to the properties of the triangles contained in the new representation.

Regardless of the test that is performed with different types of triangles, this MT of low cognitive demand, being of PWoC 02, since it requires instrumental procedures, such as measuring (approximate) angles with a conveyor, making the sum of three numbers, and checking that it is 180º (Figure 5c), or matching the adjacent angles highlighted with different colours so that they form a straight base shape, or represent a flat angle.

For checking the second type of triangle, an explicit way of solving the task is suggested, since it is a replica of the first task. By doing this five times in a row, it simply reinforces the path to follow for the student to complete the task, which implies a low-level task, and the student can project the response to be obtained, without having to perform it with concrete material.

MT 2. After performing other MTs in which the relationship of the sides is modelled in *GeoGebra* based on triangles with sides of different lengths, the teacher proceeds to show the task in which he uses concrete material (rods of different sizes). In this MT, the analysis of the triangular inequality is proposed, given by the relationship between the sides of the triangle, in which the sum of two of its sides will always be greater than the third, as proposed in Figure 5.

Figure 5

Representation of a Triangle Built with Rods. (Adapted from class: 'triangular properties in Colombian Sign Language live mathematics class' of the educational channel INSOR (https://youtu.be/DEaw2yKL10c)

a.

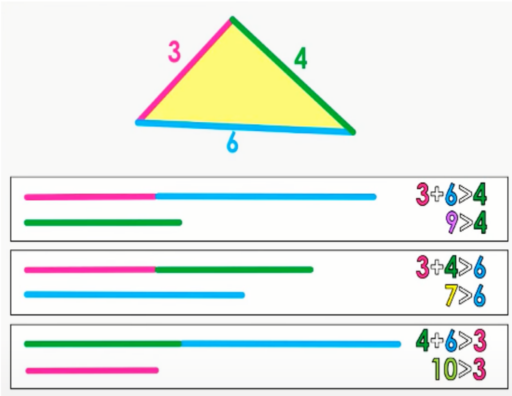
b. 

Figure 5a shows a triangle constructed with sides, five units, three units, and six units, in which they meet the ratios: 5 +3 >6, 5+6>3 and 3+6>5. After that, two (2) examples in which the inequality is not met are presented, as is the case of the triangles of two units, three units, and six units, where the sum of two of their sides (two units and three units) is less than the third side and does not reach to form a geometric figure (closed). Also, the next triangle uses two units, three units, and five units, where the sum of two of its sides units and three units is equal to the third side (five units).

The three examples used with the specific material invite participants to inquire about the possibility of making comparisons between the sides without explicitly stating the relationship between them. In the fourth example (Figure 5b), the relationships of triangular inequality are enunciated, with symbolic representations of the value of the sides, in such a way that each colour used on one side of the triangle represents its value in the approach of inequality.

By analysing the expected learning limitations for this task, three limitations can be identified: a. those related to the representations of inequality, b. the passage from the inductive to the deductive, and c. the relationships between the magnitudes involved.

This task evidences a lack of resources to promote the investigation and verification of the fulfilment of inequality for another group of numbers that represent the value of the sides. This corresponds to the limitations related to learning the concept of a triangle based on the representation-model relationship, or prototypical examples (Climent, 2002) and, with the obstacle caused by a small group of examples (Montes, Climent, and Contreras, 2022).

This is to reach the generalisation of what is intended of inequality. We could use resources as other representations and questions with other approaches that include values of the sides of a triangle that meet the condition of triangular inequality (not necessarily with natural numbers). Also, give a new example (with material or numbers representing the values) and wait for participants to confirm whether they will meet the triangular inequality. To the latter, it is necessary to identify the implications of the lack of an effective interaction, which does not allow to develop from the reading of the participants' messages, since the participation was null (Cf. https://youtu.be/DEaw2yKL10c [51:52 - 11:03:05]).

This task does not transcend the enunciation of triangular inequality as a symbolic representation due to the heterogeneity of the population participating in the class and is limited to the use of number representation and colour equivalence (Figure 5b). This refers to the need to integrate elements of triangular inequality to promote problem solving from the modelling of situations, in which it approaches students to new concepts to carry out inductive and deductive processes as appropriate (Rico, 2009). This can become a difficulty later, as they may not recognise the generalisation of the occurrence in any value of sides a, b, and c, as reported by Rosich et al. (2006).

According to Porras (2013, p. 6), triangular inequality presupposes the establishment of relationships between the parts that make up inequality, that is, the value of a pair of sides concerning the rest, without necessarily implying deductive reasoning. It is possible that, with this MT, the student can apply this situation in multiple numerical sets, for the value assigned to the magnitude of the length of the sides, i.e., he/she can make the transfer of his/her deductions to the Real numbers ($\mathbb{R}^+$). However, the author presents a small number of examples that will not allow generalisation (Montes, Climent, & Contreras, 2022). Examples are limited to the recognition of lengths with integer values. That is, making some questions about other values, even higher than those used with the lengths of the concrete material used, will promote the realisation of conjectures about the concept, to identify whether the students understood the essence of said inequality and the construction of the triangle.

Above all, the passage of prototypical figures (Gutierrez, 2013) to other proposals by the same participants will be guaranteed, or perhaps, to propose questions such as the verification of the fulfilment of inequality for groups of three numbers that meet the condition of being the value of the side of the right triangle. This would allow us to recognise that teaching geometry, in addition to developing spatial thought, allows us to establish arithmetic and algebraic relationships based on geometric constructions (Gutiérrez, 2002), this being an adequate pretext to transcend the basic levels of the participants' notions (deaf and hearing), and lead them to characterise geometric concepts through the properties of numbers.

This MT is a procedure without connections (PWC03), where representations with manipulative materials are usually used, and several examples where triangular inequality is met and where its cause is not met are reviewed (open figure or overlapping sides that fail to tilt to form a closed figure). We concluded that, although the examples used to give meaning to triangular inequality help understanding said approach, we can use questions to allow transcending the task towards generalisation with other numerical sets the deaf population will find with difficulties, due to the establishment of relationships between the concept of triangular inequality and its representations (Rosich et al., 2006).

MT 3. The class begins with a contextualisation of the algebraic representation of the Pythagorean theorem. Subsequently, five different ways of checking the Pythagorean theorem are modelled as presented in Figure 6. In this MT, the recognition of the algebraic representation of the theorem and its application in various situations is proposed.

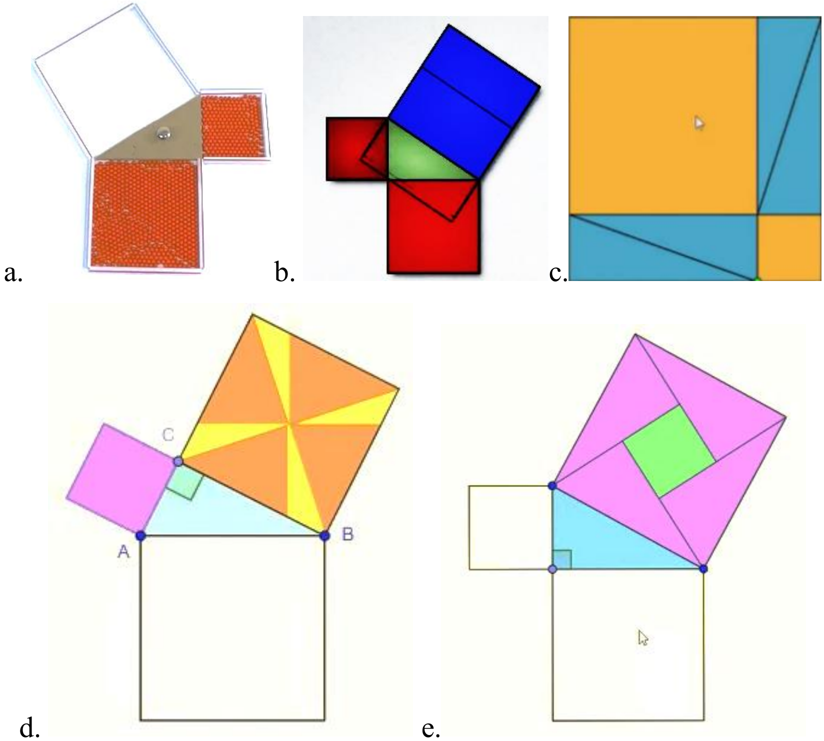
This MT aims to show the student the equivalence of the square of the sum of the sides of a right triangle with the square of the value of the hypotenuse, as shown in Figure 6.

For this, the representation made from the emptying of surfaces (Figure 6a) and the movement of the four *GeoGebra* models (Figures 6b, 6c, 6d, 6e) present the equivalence relationship between the three-square surfaces, two formed from the sides adjacent to the right angle, and the third corresponds to the square formed with the side of the hypotenuse.

Within the learning limitations analysed for this task, we identified the concept represented by the Pythagorean theorem, the use of geometric representations, and the poor understanding of the demonstration.

Figure 6

Pythagorean Theorem Check. (Adapted from class: 'Pythagorean Theorem in Colombian Sign Language live math class' of the educational channel INSOR, https://youtu.be/3b875GTTxNg)



The presentation of the Pythagorean theorem is reduced to the representation of it as the sum of the square of the cathects is equal to the square of the hypotenuse, whose demonstration usually appears reduced to graphic representations (Cañadas, 2001). This corresponds only to an expression of the theorem, so "mathematical concepts cannot be confused with the way of representing them" (Troyano & Flores, 2016, p. 55). In particular, the treatment of the theorem for this concept is reduced to the repeated presentation of representations of triangular surfaces to verify longitudinal measurements (sides of the right triangle), which implies only relationships between the registers of symbolic and geometric representation, without transcending the understanding of the demonstration and the implications of the triangles presented there, which, according to Socas (2008) does not transcend the complexity of the understanding of the mathematical object.

Although different representations are proposed on "the proof" of the Pythagorean theorem, a reflection by the student that leads to the relationship between the different geometric representations of the demonstration is not promoted (Altamirano-Chavarría, 2021). Thus, there is a need to involve the participant in the use of guiding questions that lead to identifying common aspects in the different tests carried out of the theorem, according to those defined by Troyano and Flores (2016), such as: the metric relationship between surfaces of the squares (Figure 6a, 6b, 6d, 6e), the metric relationship of the lengths of the sides of the rectangle (figure 6c), and the necessary and sufficient condition for the triangle to be rectangle, which is not contained within the exemplified representations of Figure 6).

The implementation of values for each of the modelled cases, as examples of that generality, allows students to identify the variation of the value of the sides of the right triangles seen in the modelling, especially in Figure 6b, whose squares (red and blue) can have a variation in the lengths of the sides. According to Romero (2011, p. 47): "the learning of geometry must offer continuous opportunities to build, draw, model, measure, or classify according to freely chosen criteria", which implies proposing different representations but also leading to generalisations and arguments, which are necessary for the learning of mathematical contents based on chosen criteria (Gutiérrez, 2013). However, in the example of the MT3, we do not delve into the practices of mathematics, such as motivating deductive, inductive processes, generalising or conjecturing based on the models of the theorem presented (Romero, 2011, p. 46). On the contrary, the demonstration becomes repetitive, not showing clarity about the treatment of representations, which, in Gallo's (2011) words, indicate proposals of a lower level of reasoning (in contrast, for example, with proposals towards hearing populations) considering only basic cognitive processes that do not meet the needs of the deaf.

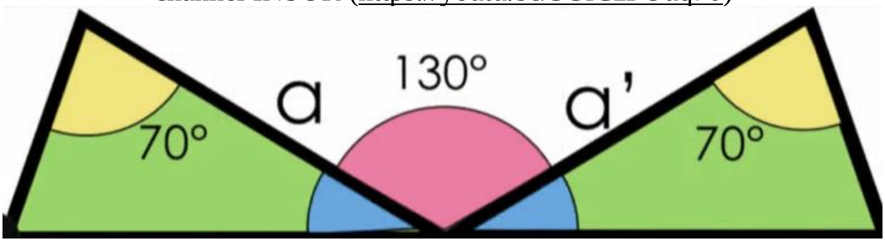
This MT is a procedure with connections (PWC02) that implicitly suggests the paths to follow, the same ones that do not allow generalising these paths from understanding the characteristics of the right triangles and the use of the theorem to find the value of one of its sides, even if it becomes repetitive. In this task, the use of graphical representations and the modelling of the variation of the sides is necessary for the fulfilment of the equivalence proposed by the theorem. Likewise, this type of task allows connections to be made with other representations and characteristics and relationships of the right triangle, which allows us to elaborate on the meaning of why equivalence is maintained. This will allow us to easily identify the application procedure in specific tasks that involve the application of the theorem for their resolution.

MT 4. This task proposes to identify the use of the Angle-Side-Angle (ALA) criterion to check whether the pair of triangles are congruent. The task is projected with the shape built (Figure 7).

In this MT, students should identify the congruence of two triangles by applying some of the previously presented criteria.

Figure 7

Triangle Congruence (ALA criterion). (Adapted from class: 'Congruence of triangles in Colombian Sign Language live math classes of the educational channel INSOR (https://youtu.be/3GlCzPOuq70)



This MT aims to show the student the characteristics of the sides, external angles, and adjacent angles to relate them to the criterion of congruence between triangles: angle-side-angle. Figure 7 shows the representations used in different colours with which they intend to relate equivalent angles between the two triangles and indicates the surfaces of the triangles in green. For both triangles, we have an equivalent pair of angles of 70 °, highlighted in yellow, and a pair of angles adjacent (blue) to a common angle (red).

The notation of the angles has been made as a circular sector that denotes surface and not the measurement of the opening between two segments. The event matches the three types of angles highlighted in yellow, blue, and red. Given this, and if the portion highlighted in blue, added to the common red angle, whose value is 130º, we deduce that the angles are equivalent, and its value will be 25º. After this, we deduced the internal angle of each triangle, since they had previously identified the value of the sum of the internal angles of the triangle (180º), for which they obtained the angle that has not been highlighted and that is part of both triangles, has a value of 85º. Likewise, to deduce the congruence of both triangles, the equivalence of the sides a and a' is highlighted, for which it is deduced that both triangles are congruent.

By analysing the expected learning limitations for this task, it is possible to identify the limitations related to the repetition of exercises based on the criterion explained; the reduced field of examples used with the criteria, and the non-presentation of examples in which the exposed criteria are not met. The class sequence shows that after the application of each criterion is exemplified, therefore, the example that is projected has to do with the criterion just explained. Each of the examples presented could have been developed in reverse, in which one of the examples did not meet the criterion explained above, thereby promoting mental flexibility when questioned about their knowledge and what knowledge would be needed for the development of the task. The use of predetermined forms, and the lack of records during the explanation, do not allow us to recognise the common characteristics of both triangles in which predominate articulated resources with three registration systems: graphic, symbolic, and visual-gestural (Calderón & León, 2016), which favour the understanding of these concepts.

As in MT1 (Figure 4), the angle is represented as a surface (yellow in Figure 7) in which it stands out in both cases in the same colour. In addition, in all the examples developed, triangles appear with colours intentionally placed to make it easier to identify angles and, consequently, congruence. This reveals the simplification of the concept (Gallo, 2011) by offering a reduced set of examples, which do not allow more elaborate reasoning about what is presented (Montes, Climent, & Contreras, 2022), which prevents generalisation. In addition, there are no triangles in which the criteria are not met. So, it is not easy to identify whether the student will understand the congruence criterion in the absence of the intentionally given colour or highlights.

Geometric figures have not been called by their vertices, which complicates the denotation of angles or sides, limiting them to the representations of the parts of the figures highlighted as surfaces or segments of colours as a strategy of simplification of statements or concepts to make them understandable (Rosich & Serrano, 1998). This has consequences in understanding the symbolic representation registers of the angles, the equivalence symbols between angles and sides, and the formalisation of the deduction made, which, finally, will result in the establishment of relationships, classification, and attribution of properties (Rosich et al., 2006), which is limited to the knowledge of body language, or what is manifested by the audio description or subtitles of the live transmission.

This MT is a memorisation task (MEM01), in which the use of criteria is proposed to identify congruence; however, they involve reproducing the analysis of angles and sides of a pair of given triangles, in which characteristics are repeated as in previous examples, with other given criteria (Side-Side-Side and Side-Angle-Side). The understanding of the example and the identification of characteristics is simple, since the measurements of the sides, the characteristics of the angles, and the identification of the remaining angle, become evident because of the colouring of what one wants to be highlighted for the identification and application of the congruence criterion.

In general, the four MTs allow recognising different levels of cognitive demands presented in the theoretical framework of this article, in which simple tasks can be explored in various ways so that they can intentionally reach other levels of cognitive demand.

Finally, when analysing the cognitive demand (Smith & Stein, 2016) of four MTs, in the online classes proposed for the deaf school population of Colombia, we could identify different levels of cognitive demand (MEM, PWoC, PWC, and DM), in which MTs do not maintain a constant in the amount of MTs proposed at some level of cognitive demand, but rather pass, for example, from a PWC task to a MEM, or vice versa, unlike the concept (geometry, measurement, variation, numerical, or random). This emphasises the development of spatial thinking; none of the classes proposed or developed high-level DM-type tasks. However, these classes can be characterised through low cognitive demand MTs, since 24 of the 30 tasks proposed in the development of the six classes corresponded to low demand (15 of MEM type and nine of PWoC), the rest of the MTs correspond to PWC type tasks, concentrated in class 1, 4, 5, and 6.

CONCLUSIONS

This article has focused on the analysis of MTs for the deaf population since the proposal of INSOR. With this, it is possible to identify different approaches to geometric objects and their approach through digital tools for the deaf population.

The teaching actions incorporate various resources in favour of the student's learning. In it, the planning of a class and the selection and implementation of MTs are part of a teaching exercise that involves the learning of students through the appropriation of strategies and actions to develop mathematical thinking (Acevedo-Rincón, 2017). In addition, the appropriation of various strategies converges in the student’s actions on mathematical objects. This is an action mediated by the different moments of the class, which allows us to negotiate meanings, participate, make mistakes, return to the first step, or find a path of solution for each situation presented, among others.

According to the model proposed by Stein, Grover, and Henningsen (1996) and reformulated by the first author (Figure 1), the action is based on knowledge *of* practice (Cochram Smith, & Little, 1999; 2009). Additionally, there is evidence of knowledge *for the* practice of teaching mathematics (Ball, Thames & Phelps, 2008; Cochram Smith & Little, 2009), pedagogical and didactic knowledge of mathematical content (resources, curriculum, characteristics of learning, teaching mathematics to the population with disabilities, actions, and interactions of students against MTs, etc.).

One of the factors that lead to the recognition of the low levels of demand corresponds to the diversity of the population, since it is a class open to the school population and, sometimes, parents, teachers, or interpreters. This implies finding common points in the development of the MTs, in which most people who view the class can participate. During the development of the different moments of the class, a joint activity is assumed in two ways, from student to teacher and vice versa, which are mediated by mathematical knowledge (Acevedo-Rincón, 2018). In this sense, the INSOR proposal allows us to identify different strategies selected (Smith & Stein, 1998) and implemented in the development of the class. These strategies were used as contexts of the different mathematical tasks.

By identifying the teaching trajectory used by a live model mathematics class in the INSOR channel, we recognise MTs with distinctive characteristics, differentiated levels of the tasks that go from basic explanations to others with greater difficulty. Likewise, audio-visual aids are another clear example of the different ways of presenting information, since this is not limited to the background voice that accompanies and translates the interpreter teacher but, at the same time, situations are modelled through different schemes and graphic representations of situations or, in others, these are developed from a video that presents a situation, which, as highlighted by Smith and Stein (1998) are practices that guide productive discussions around a mathematical concept.

Particularly, in the analysis of the four MTs corresponding to spatial thinking, there are four levels of cognitive demand among them, predominating construction activities that include the visualisation of concrete material, modelling, or recreation of situations experienced by the characters (other teachers) in the class, which allow mobilising the participation of the community through the chat of the channel or the alternative communication channels, for example, WhatsApp. This kind of MT guarantees the participants' connection and the significance of the exposed procedures, or the striking materials used to develop the situations (Ignatius, Nogueira, & Da Silva, 2018).

It is noteworthy that these tasks are prepared in advance by several teachers (of mathematics and interpreters) who plan together (Fiorentini, 2013). Their contents are addressed according to the survey of "need" for explanation and/or deepening in the conceptual aspects and of the development of processes typical of the teaching-learning of mathematics such as reasoning, exercise, comparison, modelling, and/or problem solving.

Each teaching strategy incorporated into MTs allows students to interact with mathematical knowledge in various ways. Participation, as a constituent element in learning (Wenger, 2013), maintains several factors that influence the normal development of a class, conceived under the student, knowledge, and teacher interaction model. So, one of the difficulties found lies in the late participation between the answers recorded in the chat and the questions prepared by the teacher. The lateness is due to the gap between the transmission tool and the time the question takes to be answered, and the same happens with the return of the answers.

The empty seconds in the transmissions lower the levels of attention of those who are connected, which are "adjusted" with greetings to the participants, which are also used to check if there are answers or questions in the WhatsApp number enabled to generate conversation among the participants. Likewise, the MTs are expected to be resolved in the shortest possible time due to the implications in the balance between the transmission time and the participants' interaction, i.e., the longer airtime, the less efficient participants' interaction. This type of response requires a basic argumentation and the effort that this demands from the participants is minimal. Including more procedural activities with connections and/or construction of mathematics would involve limiting the number of MTs to be developed in the class, in addition to ensuring the development of other mathematical processes other than exercise.

By delving deeper into the learning of deaf students (and/or hearing participants) through online lessons, we highlighted that this type of class format of synchronous encounters with a bimonthly frequency between lessons does not allow an individual accompaniment of the participants during the synchronous session or between sessions. However, during the session, some understandings of the explanations given in groups were evident as they participated in the chat with questions or answers from the MTs, or by asking questions. This allows relating the levels of cognitive demand of the tasks with the constant participation and the manifestation of strategies during the session. However, a formative evaluation could not be implemented among the attendees since not everyone was guaranteed participation, either because they did not know some basic procedures or because competition was not encouraged in the delivery of the answers to be projected for the rest of the class. Even so, the participants' answers, when they existed, were considered and screened live (mostly during the session). Although the answers sent were not all correct, the teacher ensured feedback on the solution of the MT of class, even if they were illegible (either by the writing or by the signs used).

Likewise, the remaining MTs to be developed after each session were resolved at the end of the live sessions based on the previously selected answers of the participants. These tasks were counted as part of the reading of the six geometry thinking classes, being characterised as MT type PWoC 02 (MT1, 05/07/2019 class), PWC03 (MT2, 05/07/2019 class), PWC02 (MT3, 08/09/2019 class), MEM01(MT4, 03/10/2019 class). That is, of the six classes, four had after-school MT, and most of them (three of four) were of low cognitive demand.

This proposal of INSOR allows us to transcend the integration of the deaf person in a horizontal and symmetrical dialogue between the two cultures (deaf and hearing). This implies allowing the participation of the deaf person in different spaces, as their actions are incorporated and legitimised through different means (video, text, graphics, etc.), valued and feedbacked during the classes.

It should be noted that the production and exchange of geometric knowledge during classes is not limited to the physical space under an institutional denomination as represented by INSOR. Thus, the experiences lived from the video classes leave diverse learnings, such as the construction of spaces of symmetrical interaction where all participants (deaf and hearing) know the language to guarantee non-exclusive interactions and approach the deaf to the understanding of the spaces they inhabit and of the identity in an intercultural citizenship not limited to the communication in sign language. The speaking interpreter can constitute a bridge for all class participants; for example, people curious to know the interactions with diverse populations in which the decrease or absence of hearing prevails.

This article finally reports a rigorous analysis of tasks that develop spatial thinking and geometric systems, in which the objective of approaching the recognition, description, and analysis of MTs addressed for concepts and processes is achieved. This constitutes an invitation to the implementation of new levels of cognitive requirement in which participants are allowed to transcend the repetition of particular procedures, towards the level of doing mathematics, through timely reading of situations with connections (in mathematics, towards their everyday life, or towards other sciences) that lead to the formulation of hypotheses and allow higher levels of argumentation, not limiting responses to a number (for example, the value of the sum of the internal angles, in Figure 2) or the identification of a theorem, property, or characteristic in the development of purely mathematical situations (postulate identification, in Figure 5). This would imply a greater recognition of the (fixed) participating population, to deepen in critical issues so that students, mainly, can perform adequately in various situations, or in the 'standardised' tests, for example, to enter the university.

Finally, although the proposal is pioneering at the national level, and its broadcasts constantly support the deaf (and hearing) population of Colombia, it must continue to reflect on the possible improvements in more inclusive teaching, in which the virtual resources offered are not limited to the representations of the concepts, but transcend the recognition of characteristics, properties. Thus, from these, processes of mathematical work are promoted in students, without limiting their learning to a reduced set of options, or a reiterative exemplification that change shape or colour. This cognitive requirement for the deaf population must transcend the understanding that the hearing has about mathematical education and, instead, promote spaces for the education of the deaf population based on the needs of the Colombian population.

AUTHORSHIP CONTRIBUTION STATEMENT

JPAR performed data collection and preliminary analysis. Both authors, CEFP and JPAR, discussed the planning of the article and actively participated in the discussion of the results, reviewing, and approving the final version of the paper.

DATA AVAILABILITY STATEMENT

The data produced and supporting the results of this study may be provided by the corresponding author, JPAR, upon reasonable request.

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