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Trigonometry vs. Trigonometric Integration

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ABSTRACT

Background: Relating and applying trigonometric contents to solve problem situations involving trigonometric integration is not easy for university students. **Objective**: To recognise trigonometry as a mathematical tool with multiple applications in their professional context. **Design**: Exploratory, descriptive, analytical-interpretative. **Setting and participants:** University engineering students. **Data collection and analysis**: Two stages were programmed: preparation and formalisation. The classes were videotaped and transcribed into units of analysis in order to triangulate the information collected. **Results**: The importance of trigonometry as a mathematical tool that allows modelling different problem situations was highlighted. **Conclusions**: The development of mathematical competencies in young people was evidenced, manifested in the progress of advanced mathematical thinking skills, such as: abstracting, generalising, synthesising, defining, and demonstrating; manifested when facing different proposed situations where students understood that when using trigonometric integration, different trigonometric elements are also used, allowing for obtaining simpler expressions and facilitate integration.

Keywords: Trigonometric integration, Advanced mathematical thinking, Trigonometry, Engineering students.

Trigonometría vs integración trigonométrica

RESUMEN

Contexto: Relacionar y aplicar contenidos trigonométricos en la solución de situaciones problema que involucran integración trigonométrica no es tarea sencilla para estudiantes universitarios. En la muestra seleccionada se observó ausencia de significados trigonométricos, que no les permiten hacer conexión con el entorno y con situaciones problema propias de su formación profesional. **Objetivo**: Reconocer la trigonometría como una herramienta matemática con múltiples aplicaciones en su contexto profesional. **Diseño**: Exploratorio de tipo descriptivo, analítico-interpretativo. **Entorno y participantes**: Estudiantes universitarios de ingeniería. **Recogida y análisis**

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de datos: Se programaron dos fases: de aprestamiento y de formalización. Las clases fueron videograbadas, se trascribieron en unidades de análisis para luego triangular la información recopilada. **Resultados**: Rescatar la importancia de la trigonometría como herramienta matemática que permite modelizar diversas situaciones problema. **Conclusiones:** Se evidenció desarrollo de competencias matemáticas en los jóvenes, manifiestas en progreso de habilidades propias del pensamiento matemático avanzado, tales como: abstraer, generalizar, sintetizar, definir y demostrar, manifiestas al enfrentar diversas situaciones propuestas, donde los participantes comprendieron que, al usar integración trigonométrica, también se usan diferentes elementos trigonométricos que permiten obtener expresiones más sencillas y que facilitan la integración.

Palabras clave: Integración Trigonométrica, Pensamiento Matemático Avanzado, Trigonometría, Estudiantes de Ingeniería.

Trigonometria vs integração trigonométrica

RESUMO

Contexto: Relacionar e aplicar conteúdos trigonométricos na solução de situações problemáticas envolvendo integração trigonométrica não é uma tarefa fácil para estudantes universitários. Na amostra selecionada, observou-se ausência de significados trigonométricos, o que não lhes permite fazer uma conexão com o ambiente e com situações problemáticas específicas de sua formação profissional. **Objetivo**: Reconhecer a trigonometria como uma ferramenta matemática com múltiplas aplicações em seu contexto profissional. **Design:** Exploratório, descritivo, analítico-interpretativo, tipo descritivo. **Ambiente e participantes**: estudantes universitários de engenharia. **Coleta e análise de dados**: Foram programadas duas fases: preparação e formalização. As palestras foram gravadas em vídeo, transcritas em unidades de análise e depois trianguladas. **Resultados:** Destacou-se a importância da trigonometria como ferramenta matemática para modelagem de diferentes situações problemáticas. **Conclusões:** O desenvolvimento das competências matemáticas nos jovens foi evidenciado no progresso das habilidades de pensamento matemático avançado, tais como abstração, generalização, sintetização, definição e demonstração manifestadas ao enfrentar várias situações propostas, onde os estudantes entenderam que, ao utilizar a integração trigonométrica, também são utilizados diferentes elementos trigonométricos, que permitem obter expressões mais simples e que facilitam a integração.

Palavras-chave: Integração Trigonométrica, Pensamento Matemático Avançado, Trigonometria, Estudantes de Engenharia.

INTRODUCTION

This research encompassed five groups of engineering students from several areas attending Calculus 2, a subject that includes integral calculus and some elements of vector calculus. As a first movement, we conducted a semi-structured interview with this cohort to determine their knowledge level of trigonometry and whether they established any relationship between this branch of mathematics and their professional education. The answers showed us that most thought there was little or no relationship. Later, during the investigation, we observed that they could not relate trigonometry elements (ratios, functions, trigonometric identities) with problem situations typical of their professional formation. This finding revealed students' difficulties transitioning from trigonometry to trigonometric integration. For example, they find it complex to visualise any modelling that uses the Pythagorean theorem based on the construction of a right triangle, where the original expression defines a trigonometric function of one of its acute angles, to find some assertive solution to the proposed problem situations. Another deadlock identified is understanding that many integrals can be calculated by manipulating the integrand through trigonometric identities, i.e., when the integrals present powers of trigonometric functions, it is necessary to use different trigonometric identities to obtain a new, also trigonometric, more straightforward expression that facilitates integration. Finally, the investigation allowed us to identify that some students recognise that the most used identities are the Pythagorean and the addition and subtraction of angles but find it difficult to use them in context.

Theoretically, we used two referents, trigonometry and advanced mathematical thinking focused on trigonometric integration. In this type of thinking, students are expected to acquire a global mathematical meaning of various intertwined concepts that will help them develop mathematical competencies focused on abstracting, analysing, categorising, generalising, synthesising, defining, and demonstrating. Hence, the need to distinguish between *mathematical knowledge used* and the *application of mathematical concepts* when looking for a solution to a specific proposed situation.

The results show that students find it challenging to solve problems that combine trigonometric tools (ratio, function, trigonometric series, especially trigonometric integration) and situations typical of their speciality area. In the case of trigonometric integration, we observed difficulties identifying and solving situations involving: 1) products and powers of $sin\, x$ and $\cos x$, 2) products and powers of $tan\ \ x$ and $\sec x$, and 3) using reduction formulas to solve trigonometric integrals. Hence the importance of studying the uses given to trigonometry so that they help students to signify this branch of mathematics during their university education, where the formalisation of concepts and elements of advanced mathematical thinking weighs.

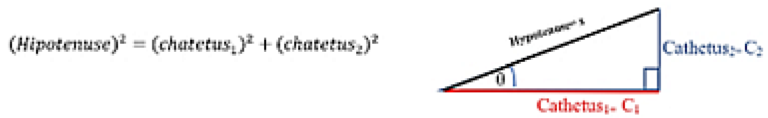
THEORETICAL REFERENCES

This work is theoretically supported by two referents: elements of trigonometry and advanced mathematical thinking, particularly trigonometric integration.

Trigonometry. Branch of mathematics that studies the relationship between sides and angles of triangles. The functions associated with these angles are called *trigonometric functions*. Among the applications is the study of spheres in the geometry of space and engineering. Geometrically, in the Pythagorean theorem, the hypotenuse and any of the legs (cathetus) are obtained as follows: $hypotenuse = \sqrt{(leg_1)^2 + (leg_2)^2}$ equivalent to $leg = \sqrt{(hypotenuse)^2 - (leg)^2}$. The mathematical and geometric representation of the theorem is shown in Figure 1.

Figure 1

Representation of the Pythagorean theorem.



A trigonometric ratio is a quotient between two magnitudes of the triangle (legs or between the hypotenuse and one of the legs), resulting in a numerical value. A trigonometric function is an application $f: \mathbb{R} \to \mathbb{R}$, which makes each real number correspond with another real number. The trigonometric functions allow us to extend the definition of trigonometric ratios to all real and complex numbers. Now, let us consider that the variable is always in the numerator of the fraction obtained as a function. Three trigonometric functions are determined: $sin\,\theta, tan\,\theta$ y $sec\,\theta$, shown in Table 1.

Table 1

Some angular functions

Notation	Definition
$\boldsymbol{Sin\ \theta}$	The ratio between the leg (leg_2) and the hypotenuse. $Sin\ \theta = \frac{Opposite\ leg}{Hypotenuse} = \frac{C_2}{h}$
$\boldsymbol{Tan\,\theta}$	The ratio between the opposite leg (C_2) and the adjacent leg (C1). $Tan\ \theta = \frac{Opposite\ leg}{Adjacent\ leg} = \frac{\boldsymbol{C_2}}{\boldsymbol{C_1}}$
$\boldsymbol{Sec\ \theta}$	The ratio between the hypotenuse and the adjacent leg (C1) $Sec\ \theta = \frac{Hypotenuse}{Adjacent\ leg} = \frac{h}{C_1}$, also known as the reciprocal of the cosine. That is, $Sec\ \theta\ \cdot Cos\ \theta = 1$.

On the other hand, a *trigonometric identity* is an equality between expressions containing trigonometric functions. It is valid for all angle values in which the functions (and the arithmetic operations involved) are defined. The Pythagorean identities and the identities for addition and subtraction of angles shown in Figure 2 are highlighted.

Figure 2

Basic trigonometric identities (own elaboration).

Pythagorean identities	Identities for angle addition and subtraction
$sin^2x + cos\ ^2x = 1$	$sin(x \pm y) = sinx\ \cdot cosy \pm siny \cdot \cos x$
$1 + tan^2x = sec^2x$	$cos(x \pm y) = cosx \cdot \cos y \mp sin\ x \cdot sin\ y$
$1 + ctan^2x = Csc^2x$	

Advanced Mathematical Thinking

The mathematics education literature distinguishes *advanced mathematical thinking* (AMT) as a line of research dedicated exclusively to mathematical concepts specific to university or “advanced” mathematics. Vinner and Herschkowitz (1980) and Valdivé and Garbin (2008) focused their research on the mental images that students evoke that conflict with institutionalised mathematical definitions[1].

Within the AMT, the student's conceptual scheme of a mathematical concept is initially described as the entire cognitive structure associated with the concept, including all mental images, properties and processes associated with the mathematical notion (Garbin, 2005; Valdivé & Garbin, 2013). Mateus-Nieves and Rojas (2020, p. 69) recognised that the AMT is typical of university education because "(...) the progressive mathematisation implies the need to abstract, define, analyse, and formalise. Among the cognitive processes with a psychological component, besides abstracting, we can highlight representing, conceptualising, inducing, and visualising". However, although "(...) abstraction is not a characteristic of higher mathematics, nor is analysing, categorising, conjecturing, generalising, synthesising, defining, demonstrating, formalising, it is clear that these last three gain greater importance in higher education courses (...)" (p.69). Therefore, in this paper a distinction is made between *mathematical knowledge used and* the *application of mathematical concepts*, understanding by "*use*" Cabañas' (2011, p. 98) definition, "the way a specific notion is used or adopted in a specific context". Both *use* and *application* can generate meanings regarding the mathematics employed. However, the difference between them is that, in *use*, there is the functionality of mathematics; i.e., it can be employed or adopted to solve problems in different contexts, where the meanings generated and their understanding are privileged. On the other hand, the *application* is generally restricted to the meanings promoted by the school mathematical discourse employed, which, from what has been observed in various class sessions, are regularly of a rote type when solving problems. In this situation, students rarely show understanding and do not seem to recognise the type of mathematics employed.

Mateus-Nieves and Font (2021) locate integral calculus within the AMT and describe three necessary global epistemic configurations to design thematic classes with greater didactic clarity for students. Mateus-Nieves and Hernández (2020) studied the global meaning of the integral that three groups of university students reach when its epistemic complexity is articulated when teaching it. Mateus-Nieves (2021) investigates the epistemology of the integral

[1]In this work, *institutionalised mathematics* is understood as the theoretical proposal of the onto-semiotic approach of cognition and mathematical instruction (Godino, 2002), where the following construct is proposed: *institutional meaning* as the content assigned to an expression that is recognised by the mathematical community as true and valid.

as an element of the AMT. Mateus-Nieves and Moreno (2021) focused on the AMT, designing and applying a didactic sequence with activities to analyse variational aspects of the function concept and how two groups of students attending Precalculus interpreted them. They demonstrated that, by engaging in the activity, students built successful approaches to understanding and using functions as models of situations of change and processes of variation.

On the other hand, within the AMT, an integral is called *trigonometric* when the integrand consists of *trigonometric functions and constants*. With the selected sample, this work formalised that, for the resolution of problem situations both in context and in specific exercises, the integration methods and theorems must be used considering the suggestions described in Table 2.

Table 2

Hints for calculating trigonometric integrals

1.	Use a trigonometric identity and simplify it when trigonometric functions are present. If possible, use a graph that models the situation.
2.	Try to remove a square root. It is usually done after completing a square or a trigonometric substitution.
3.	Reduce an incorrect fraction[2].
4.	Separate the elements of the numerator from the denominator of the fraction.
5.	Multiply by a unit form $\frac{g(x)}{g(x)}$, which, when multiplied by the integrator $f(x)$, allows adequately modifying $\frac{[f(x)g(x]}{g(x)}$.
6.	Try to substitute $f(x)$ with $\frac{1}{f(x)}$. Have a table of trigonometric identities at hand and make the suitable substitutions until you reach the "basic formulas".

METHODOLOGY

This research is exploratory, descriptive, and analytical-interpretative. Exploratory because it seeks to facilitate the understanding of the topic raised; descriptive and analytical-interpretative because it seeks to specify properties and characteristics students show when faced with trigonometric integration. Cohort: we selected 150 students from various engineering careers at a non-state university. We organised two activities to select the sample: a written test consisting of five problem situations and five specific exercises on right

[2]That is, to recognise the whole part in an improper fraction.

triangles and a semi-structured interview to identify students' mathematical knowledge and the level of appropriation of trigonometric concepts and whether they saw any relationship with their professional education. The results were systematised and triangulated according to the agreed criteria (see Table 3, results section).

The class sessions were recorded and transcribed into units of analysis. The researcher observed and wrote in a field diary the data considered relevant for the development of the investigation. The units of analysis were triangulated with the data from the field diary. This information allowed for detailed monitoring of students' performance and results and conclusions.

The research was conducted in two distinct stages. During the initial stage, called *preparation*, we reviewed and subsequently reconstructed students' pre-existing concepts because while we expected them to handle them well, we noticed shortcomings when students actively engaged with them.

Such was the case of the trigonometric identities shown in Figure 2 or the construction of polar, cylindrical, and spherical coordinates exposed in the results section. During the second stage, *formalisation*, we carried out the mathematical modelling of twelve problem situations typical of engineering education. In some, we had to sketch conventional and unconventional surfaces using mathematical definitions typical of vector calculus, specifically in performing parameterisation of surfaces based on spherical coordinates. For this, we used online GeoGebra to support the visualisation, analysis, categorisation, generalisation, and synthesis of the situations so that students could define which functions, identities, and transformations should be used to face a possible assertive solution. Unfortunately, we will not present all the situations due to space limitations. Instead, we will bring the three that can offer the most didactic elements. Also, we cannot detail two situations related to surface parameterisation using spherical coordinates.

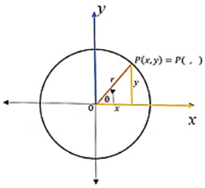
The results and conclusions sections are focused on two axes that were addressed from three problem situations that required *mathematical modelling* to identify the *use of trigonometric integration* for its solution as an emergent transition from trigonometric concepts. The first axis is because engineering students model to design and analyse the work of different situations typical of their professional work. The second is to identify research results that contribute to didactics, particularly from the AMT, which help us understand how this mathematical knowledge typical of higher education is articulated with engineers' education.

RESULTS AND ANALYSIS

Table 3 shows some criteria used to select the sample after systematising the written test results and triangulating them with the semi-structured interview.

Table 3

Sample selection criteria

Mathematical knowledge used	Application of mathematical concepts
[The student] recognises trigonometric ratios	Uses trigonometric ratios to solve problem situations.
Identifies the trigonometric ratios in a right triangle.	Gets the value of the trigonometric ratios for a given angle.
Recognises trigonometric functions.	Correctly uses trigonometric ratios to solve a right triangle.
	Uses the trigonometric functions correctly to solve problem situations.
Identifies the polar coordinates of point P by considering the following graph: Note: Starting from the angle $\boldsymbol{\theta}$ and the straight *r*, you get the point *P*, whose cartesian coordinates are *(x,y)*.  	Identifies the x-axis of the plane as the polar axis.
	Correctly recognises the polar coordinates of point P.
	Identifies the polar coordinates of point P. Correctly writes the polar coordinates of point P.
Identifies the trigonometric functions on the trigonometric circle.	Correctly employs trigonometric ratios to solve a right triangle.
	Draws the angular functions on the unit circle.
	Recognises the sign of the functions in the quadrants.
Obtains the Pythagorean identities by reasoning about the trigonometric circle.	Correctly uses the Pythagorean identities.
	Has the algebraic ability to solve the different trigonometric functions from the Pythagorean identities correctly.
Recognises the identities for multiple angles.	Knows and correctly uses identities to double angles.
	Knows and correctly uses identities to triple angles.

	Knows and correctly uses identities to medium angles.
Recognises trigonometric functions.	Correctly uses inverse trigonometric functions.

The results obtained from Table 3 allowed us to find 50 students whose performance was categorised as medium-low. We organised two groups and named the participants E1-E50 to monitor individual performance exhaustively. With this sample, we began the first stage, *preparation*. In this stage, we had to reconstruct previous concepts students should already know. We showed them that we could find other identities through simple algebraic processes from the registers in Table 1 and Figure 2, this time for *double angles.* Due to the limited space in this article, we present only those related to $sin\ x$ or $\cos x$ (see Figure 3) functions, avoiding presenting the algebraic processes, as they are considered too elementary.

Figure 3

Derived constructions

Identities for double angles	Different expressions for the functions $sin\ x$ and $cos\ x$
	$cos^2x = \frac{1+\cos 2x}{2}$ or else it is, $sin^2x = \frac{1-\cos 2x}{2}$
$\sin(2x) = 2\sin x \cdot \cos x$	$\sin x \cdot \cos y = \frac{1}{2}[\sin(x+y)+\sin(x-y)]$
$\cos(2x) = cos^2x - sin^2x = 2cos^2x - 1 = 1 - 2sin^2x$	$\cos x \cdot \cos y = \frac{1}{2}[\cos(x+y)+\cos(x-y)]$
	$\sin x \cdot \sin y = \frac{1}{2}[\cos(x-y)-\cos(x+y)]$

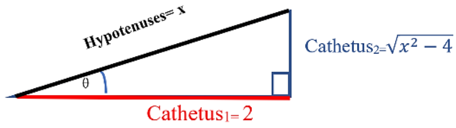
The transition of those equalities from the identities for addition and subtraction was not an easy task for the students, even though they had probably seen those themes in previous courses. Given the instrumental nature of the contents, this fact allowed us to infer that they did not handle algebra well (mathematical knowledge). We believe that students' meanings were insufficient since they could not connect the ways to employ a specific notion in a specific context. Such limitation restricts the application of mathematical concepts. Furthermore, some students thought the constructions were independent. For example, E21 stated: "*When I learned them* [the constructions], *the professor taught them as axioms; he never made constructions like the ones we see today in this class, which allows me to* *understand them as coming from algebraic processes*." [unit of analysis 96]. This speech shows that students have mathematical knowledge but cannot *apply these concepts* due to unplanned teaching, which can lead students to present epistemological obstacles. In terms of Barrantes (2006), following Brousseau, from a didactic point of view, this shows the importance of considering the elements at stake between teaching and learning. To Brousseau, "the error is not only the effect of ignorance, uncertainty, but the effect of previous knowledge, which, despite its interest or success, now turns out to be false or simply inadequate" (Barrantes, 2006, p.3). In other words, in the study of epistemological obstacles, we sought to determine the causes of errors since students' mistakes do not occur only due to their lack of prior knowledge to analyse a specific situation; sometimes, previous knowledge hinders access to new knowledge because it may be so ingrained, that students' minds may offer resistance; thus, they may find it hard to reconsider or change it.

In the second stage, *formalisation,* we institutionalised that the process of integration by trigonometric substitution enables to combine functions containing algebraic expressions that can be converted into the forms: $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$. This method eliminates the square root through right triangles, the Pythagorean theorem, and trigonometric identities. Then, we carried out mathematical modelling on the exercises and situations for students to visualise and connect the proposed situation with already-known mathematical expressions: ratios, trigonometric, and integral functions. When trying to solve them, students identify that integration is necessary but find it challenging to identify and use the necessary method(s) to solve them. Once again, this aspect led us to infer students' difficulties relating the mathematical knowledge learned to apply the concepts to particular situations.

For example, we presented the case of the integral $\int \frac{1}{x^2-4}$ dx, where they had to apply the method by trigonometric substitution. We had to guide students to recognise that this situation can be related to the Pythagorean theorem, in which it is possible to structure a right triangle where the original expression defines some trigonometric function of one of its acute angles. For the modelling, we created Figure 4, a right triangle, to identify which corresponding trigonometric function allows students to interpret the expression $x^2 - 4$, in such a way that the root of the minuend is the hypotenuse (*x)* and the root of the subtrahend one of the legs (which, in this case, is 2).

Figure 4

Interpretation of a right triangle



In this model, the appropriate trigonometric function, according to the location of the data, is $sec\ \theta$, since the variable x remains in the numerator. We completed the triangle to determine the value of leg2, considering Leg2=$\sqrt{x^2-4}$. Subsequently, we found that $\sec\theta = \frac{x}{2}$; by isolating the variable $x = 2sec\theta \rightarrow dx = 2sec\theta \tan\theta$. Substituting x and its differential in the original integral, we have the equivalent:

$$\int \frac{1}{x^2-4}\,dx = \int \frac{2\sec\theta\, tan\theta}{4sec^2\theta - 4}\,d\theta = \int \frac{2\sec\theta\, tan\theta}{4(sec^2\theta - 1)}\,d\theta$$

$$= \frac{1}{2}\int \frac{sec\theta\ tan\theta}{tan^2\theta}\,d\theta = \frac{1}{2}\int \frac{sec\theta}{\tan\theta}\,d\theta$$

now, as $sec\theta = \frac{1}{\cos\theta}$ and $tan\theta = \frac{sin\theta}{cos\theta}$, so $\frac{sec\theta}{\tan\theta} = \frac{\frac{1}{cos\theta}}{\frac{sin\theta}{cos\theta}} = Csc\theta$, in the integral, we have:

$\int \frac{1}{x^2-4}dx = \frac{1}{2}\int Csc\theta d\theta = \frac{1}{2}ln(Csc\theta - cotan\theta) + C$. As the integral should be expressed in terms of the variable x, we consider that $Csc\theta = \frac{x}{\sqrt{x^2-4}}$ and $Cotan\theta = \frac{2}{\sqrt{x^2-4}}$. Therefore, the integral was given as:

$$\int \frac{1}{x^2-4}\,dx = \frac{1}{2}\ln\left(\frac{x}{\sqrt{x^2-4}} - \frac{2}{\sqrt{x^2-4}}\right) = \frac{1}{2}\,ln\left(\frac{x-2}{\sqrt{x^2-4}}\right) + C \triangleq$$

This situation was not at all conventional for many students. For example, E5 says: "*...it turns out to be a very elaborate process for someone just learning the topic*" [unit of analysis 315]. During the conversation between the researcher and the group of students, the student added, "*It is not easy to* *achieve that ability to interpret and relate already-known mathematics with new situations*" [unit of analysis 318], which allows us to ratify students' difficulties to relate *the already-studied mathematical knowledge* to the *application of those concepts* at the intra and extra-mathematical levels. It is difficult for them to visualise the emergence of related mathematical topics, as they see them as independent mathematical entities. In terms of Doumas and Hummel (2005), it corresponds to difficulties with the development of relational thinking, referring to the ability to form and manipulate relational representations, which allows them to achieve the ability to create analogies between apparently different objects or events, and the ability to apply abstract rules in known situations. The researcher was struck by E5's affirmation: "*it turns out to be a very elaborate process…*" and asked him *why he thought* so, to which the student replied: "*During high school and in these two university years, it is the first time that a teacher builds a concept from a problem situation relating it to previous knowledge. The teachers always gave us a problem and formulas, but I never knew where or how they had been built. Many times I wondered, is that true? Who dedicated themselves to studying that? and why?*" [units of analysis 315-318]. This statement allows us to infer that this young man faced formal-mechanistic teaching processes in which the epistemological vigilance of the taught knowledge was neglected.

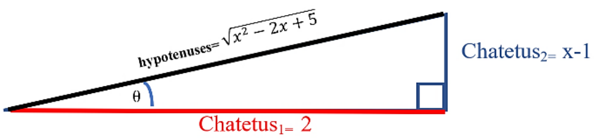
Another exercise that caused difficulties for the students was: from $\int \frac{1}{(x^2-2x+5)^2}dx$, they had to locate the values of the legs and the hypotenuse. The modelling created from a right triangle, which, at the end of the exercise, generated Figure 5, was not as easy for them as we initially thought. They had a hard time identifying that the algebraic expression must be of the form $\sqrt{x^2+a^2}$ to calculate the integral. And that this algebraic expression had to be transformed to identify the requested values and locate them in the triangle. We asked the students which algebraic process could be applied to the denominator to transform it into one in the indicated way. Some students presented the expression $\int \frac{1}{(x^2-2x+5)^2}dx = \int \frac{1}{(x^2-2x+1+4)^2}dx = \int \frac{dx}{((x-1)^2+4)^2}$ [unit of analysis 350], where the denominator is x^2+a^2. We highlight here the mathematical knowledge some students employed and the functionality of mathematics manifested in adequate algebraic handling and the commutativity in the expression presented, which leads us to infer privilege in the meanings the students generated, and their understanding. We must note that some students disagreed with this algebraic treatment because the radical did not appear. This makes us believe that these students present a *mathematical knowledge* weakness that prevents them from using and adopting the acquired knowledge to solve problems in different contexts. For them, the *application* is limited to the meanings promoted by the short-term memory-type school mathematical discourse, which confirms the importance of breaking with the formal-mechanistic paradigm of the rote learning traditionally applied in higher education.

When this difficulty was overcome by the researcher's intervention, the following was defined: $x - 1 = 2tan\theta \rightarrow 2\,sec^2\theta d\theta$ by substituting into the integral, we get: $\int \frac{1}{(x^2-2x+1+4)^2} dx = \int \frac{2\,sec^2\theta d\theta}{(4\,tan^2\theta+4)^2}$, factoring 4 into the denominator, applying the second Pythagorean identity shown in Figure 2 and simplifying, we obtain $\int \frac{2sec^2\theta}{16\,sec^4\theta} d\theta = \frac{1}{8}\int \frac{d\theta}{sec^2\theta} = \frac{1}{8}\int cos^2\theta d\theta = \frac{1}{16}\int(\cos 2\theta + 1)\,d\theta.$. Using the second identity for double angles shown in Figure 3, we have: $\frac{1}{16}\cdot\frac{1}{2}sin\;2\theta + \frac{1}{16}\theta + c,$ equivalent to: $\frac{1}{16}(sin\theta cos\theta) + \frac{1}{16}\theta + c[*]$.

As $x - 1 = 2tan\theta,$ hence $\tan\theta = \frac{x-1}{2} \leftrightarrow \theta = tan^{-1}\frac{x-1}{2}$. With these data, it is now possible to visualise the values of the legs and the hypotenuse on the right triangle.

Figure 5

Construction of the situation



From Figure 5, from which we deduced that $sin\theta = \frac{x-1}{\sqrt{x^2-2x+5}}$ and $\cos\theta = \frac{2}{\sqrt{x^2-2x+5}}$, by substituting these values into the equality marked as $[*]$, we have:

$$\int \frac{1}{(x^2-2x+5)^2}dx = \frac{1}{16}\left[\frac{x-1}{\sqrt{x^2-2x+5}}\cdot\frac{2}{\sqrt{x^2-2x+5}}\right] + \frac{1}{16}tan^{-1}\frac{x-1}{2} + c. \triangleq$$

Equivalent to

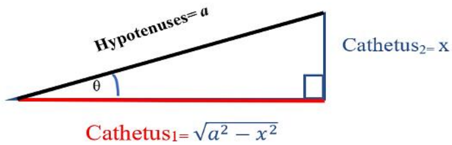
$$\int \frac{1}{(x^2-2x+5)^2}dx = \frac{1}{8}\frac{(x-1)}{x^2-2x+5} + \frac{1}{16}tan^{-1}\frac{x-1}{2} + c. \triangleq$$

Given this situation, we highlight that the difficulties presented by the students are related to not remembering the three cases studied in previous classes (mathematical knowledge used). Here the teacher in charge must recognise that both use (functionality) and application can generate meaning in the students through the mathematical discourse and the mathematics utilised. Therefore, we had to reinforce the three cases studied from the models presented in Figures 6, 7 and 8. In their order, they were:

1. If the function to be integrated contains an expression of the form $\sqrt{a^2 - x^2}$, the substitution should be $x = a\ sin\theta\ \left(con\ a > 0\ y - \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$, given that it will remove the square root.

Figure 6

Case 1



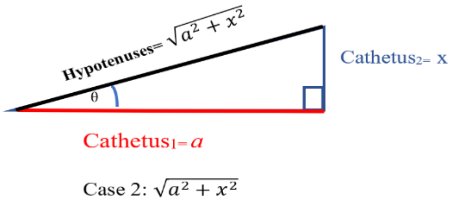
Case 1: $\sqrt{a^2 - x^2}$

By graphing on the triangle and making the substitution, the student visualises that $\sqrt{a^2 - x^2} = a\ cos\theta$.

2. If the function to be integrated contains an expression of the form $\sqrt{a^2 + x^2}$, the substitution is $x = a \tan\theta$, with $\left(a > 0, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$.

Figure 7

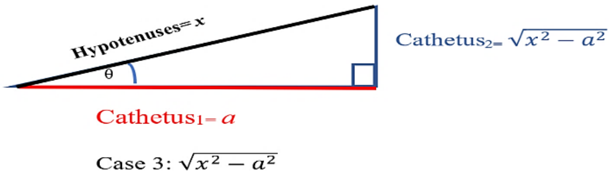
Case 2



3. If the function to be integrated contains an expression of the form $\sqrt{x^2 - a^2}$,, the substitution should be $x = a \sec \theta$,with $\left(a > 0,\ 0 \leq \theta \leq \frac{\pi}{2},\ \ o, \pi \leq \theta \leq \frac{3}{2}\pi,\ \right)$, given that it will remove the square root.

Figure 8

Case 3

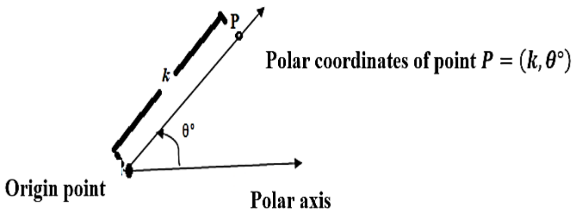


By overcoming this difficulty and exposing them to the other chosen situations where they had to relate spherical and cylindrical coordinate systems, we observed that strengthening the previously exposed relationships allowed them to obtain an approximation to the Cartesian coordinate system, recognising that they provide a simple way of describing the location of points in space. Taking advantage of this situation, we explained that some surfaces could be challenging to model with equations based on the Cartesian system, a familiar problem in engineering since, in two dimensions, polar coordinates regularly provide a sound alternative system for describing the location of a point in the plane, particularly in cases involving circles. Now, it is possible to describe the location of points in space in two different ways (cylindrical and spherical coordinates). The former helps with problems involving cylinders, e.g., calculating the volume of a round water tank or the amount of fluid that passes through a pipe. The second helps with problems related to spheres, e.g., finding the volume of vaulted structures. Faced with this issue and given the students' difficulties in understanding the situations raised, we reviewed some basic notions that had not been considered in stage 1 because we took for granted that students knew them well and could handle them: polar, cylindrical, and spherical coordinate systems and parametric surfaces. So, we followed the approach presented in Stewart (2012), as stated below.

To expand the traditional Cartesian coordinate system from two to three dimensions, it is enough to add a new axis to model the third dimension considering that a Cartesian coordinate system is defined by two orthogonal axes in a two-dimensional system and three orthogonal axes in a three-dimensional system that intersect at the origin (0,0) and (0,0,0) respectively. Therefore, to represent the points in the Cartesian plane, we employed the system of *polar coordinates*, where it is necessary to know an angle (θ) and a distance (k).

Figure 9

Polar coordinates of a point (own adaptation from Stewart, 2012, p. 639).

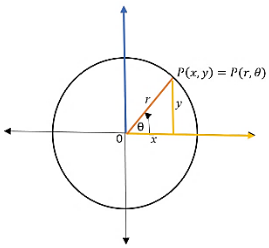


By following a process similar to that used in two dimensions, through polar coordinates, we created a new three-dimensional coordinate system called a *system of cylindrical coordinates*, indicating that cylindrical coordinates provide a natural extension of polar coordinates to three dimensions. It seemed like a simple process, but for the students, it was not. Among the difficulties identified while constructing the *system of cylindrical coordinates* as a natural extension of polar coordinates to three dimensions, E17 asked: "*Professor, can this process continue indefinitely?*" [unit of analysis 530], which reveals that this student is unaware that it is only possible to make graphs up to three dimensions.

Regarding the construction of cylindrical coordinates[3], initially, we had to declare that, from the Pythagorean correspondence established for a right triangle, the relationship between polar and Cartesian coordinates is given by the equalities: $x = r\cos\theta$, and $y = rsin\theta$, where it is possible to verify that $r = \sqrt{x^2 + y^2}$, and the formula for the polar angle is $\theta = cos^{-1}\left(\frac{x}{r}\right)$, considered in Figure 10.

Figure 10

Polar coordinates in the plane



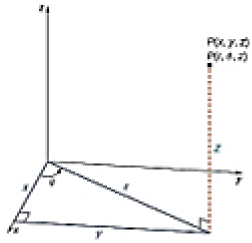
Now, in Figure 11, we identified a right triangle on the plane xy, the length of the hypotenuse is r, and θ is the measure of the angle formed by the positive x-axis and the hypotenuse. The coordinate z describes the location of the point above or below the plane xy. At this point, it was essential for the students to make the construction in GeoGebra, where they could identify the "key" to transform cylindrical coordinates to Cartesian, or rectangular and vice versa.

[3] In this system, a point in space (Figure 8) is represented by the ordered triple (r, θ, z), where

- (r, θ) are the polar coordinates of the projection of the point on the plane xy
- z is the usual coordinate z in the Cartesian coordinate system.

Figure 11

Representation of a point in space



During the conversion from rectangular to polar coordinates in two dimensions, we emphasised that the equation $\tan\theta = \frac{y}{x}$ has an infinite number of solutions. However, by restricting $0 \le \theta \le 2\pi$, it was possible to find a unique solution based on the xy quadrant of the plane at which the original point lies (*x, y, z*). We considered that if $x = 0$, then the value can $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, or 0, depending on the value of y. We observed that very few students noticed that these equations derive from the properties of right triangles. This allows us to infer that, for them, relational understanding, defined as the ability to identify operations and relate algebraic expressions in a flexible way (mathematical knowledge), is limited to a conceptual network that does not correspond to some mathematical concepts and the ability to use them to find answers or make value judgments about the reasonableness of the assigned use. So, they cannot determine what type of relationships are taking place and whether they are adequate. That is why the significance reached makes the application of mathematical concepts limited. Regarding spherical coordinates [4], we emphasised that a surface in space is represented by a set of parametric equations described in a vector function.

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[4] Understood as another generalisation of the polar coordinates of the plane when rotated about an axis. This situation presents three constitutive elements:

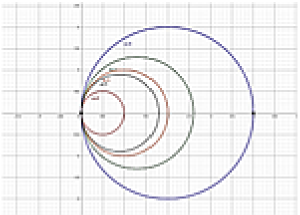
- The *radial* $r$coordinate: distance to origin
- The *polar coordinate* θ: the angle that the position vector makes with the axis.z
- The *azimuthal coordinate* φ: the angle that the projection onto the plane xy forms with the x -axis.

The variation ranges of these coordinates are: $r \in [0, \infty)$; $\theta \in [0, \pi]$; $\varphi \in (-\pi, \pi]$. It should be noted that the angle φ can also vary in the interval$[0, 2\pi)$.

Given this situation, we proposed that students find the area of the circumference $(x-a)^2 + (y)^2 = a^2, a > 0$: What can you conclude when employing trigonometric substitution and then polar coordinates? Given this situation, we noticed that when building the graph on GeoGebra online, they identified that a family of circumferences tangent to the origin of the plane was generated (Figure 12).

Figure 12

Family of circumferences



When asked about the work to be done according to the graph that the software offers, the students' proposals were almost uniform. They identified that the area was determined by the integral $A = 2\int_0^{2a} \sqrt{a^2 - (x-a)^2}dx$, and that it is a case 1 exercise, which led them to substitute the form $x = a\,sin\theta$ to remove the square root. What was difficult for the students to identify was the correct substitution. E25 proposed: "$x^2 = sin\theta$". When asked: "*What is the differential? And, by making the respective substitution in the integral, is it easier to calculate*? [unit of analysis 564], he replied: "*The differential is* $dx = \frac{cos\,\theta}{2x}$*, but I don't see whether this helps solve the exercise*" [unit of analysis 565]. This allows us to infer that the subgroup where this student worked has unclear mathematical knowledge. Therefore, the relational thinking they handle does not allow them to assertively identify the application of concepts.

After examining this proposal with the entire group, E41, a member of another subgroup, with his notebook in hand, centred on the table and proposed:

> *We worked trying several options; the one that worked best for us was:*
>
> *Let be* $x - a = a\,sin\theta$*; the differential is* $dx = acos\theta d\theta$*. Now, since it is a circumference, then* $x = 2a$ *, leading to* $a =$

$a\, sin\theta \to sin\theta = 1$, which allows us to conclude that $\theta = \frac{\pi}{2}$. And as $x_1 = 0 \to -a = a\, sin\theta,$, it means that $sen\theta = -1,$, therefore $\theta = -\frac{\pi}{2}$ which represents the limits of integration [units of analysis 578-580].

Then, E28, a member of the same subgroup, said:

We have already calculated the integral; *professor, can we go to the board*?" and writes: "$A = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 - a^2 sin^2\theta},\ a\cos\theta d\theta$ *, which is equivalent to* $2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^2\theta d\theta = 2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta = \pi a^2$, *this number represents the value of the requested area* [units of analysis 581-583].

Regarding the use of polar coordinates, the subgroup of student E30 stated:

We did this work: be $x = r\cos\theta$ and $y = r\, sin\theta$, as $(x-a)^2 + y^2 = a^2 \to x^2 + y^2 - 2ax = 0$ so, $r^2 - 2ax\, cos\theta \to r(r - 2a\, cos\theta) = 0$, then you have that $r = 2a\, cos\theta$.
Now, if we consider $A = \frac{1}{2}\int_a^b |f(\theta)|^2 d\theta = 2\left[\frac{1}{2}\int_0^{\frac{\pi}{2}} (2a\, cos\theta)^2 d\theta\right]$ you have:$= 4a^2\int_0^{\frac{\pi}{2}} \frac{1+cos\, 2\theta}{2} d\theta = 2a^2\int_0^{\frac{\pi}{2}} (1 + cos\, 2\theta) d\theta$, which, when calculating, gives: πa^2 [units of analysis 592-594].

In both productions, we observed that students treated the exercise data suitably, meaning they developed relational thinking that they expressed in understanding, ordering, and classifying the data of the proposed situation with their schemes, connecting ideas exhaustively to extract the correct answer. Thus, the mathematical knowledge was adequately used. For those students, as in Meel (2003), when citing Skem (1978), there is a distinction between relational and instrumental understanding, leading to contrasting relational thinking with procedural thinking. Following this author, we identify students, understanding when "they know what to do and why" (p. 9), i.e., they do understand (functionality of mathematics). At the same time, they use instrumental comprehension to identify "particular rules and know how to apply them" (p. 9).

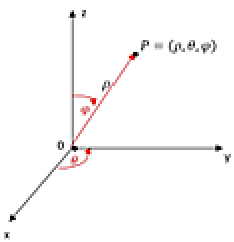
We observed that the GeoGebra Surface command syntax allowed them to link the mathematical definition with its graphical representation, viewed in 3D, which makes us believe that this software offered the students a user-friendly environment. They interacted with the commands straight through the input bar, which allowed them to visualise the graph of unbounded surfaces from different angles. This technological environment became a tool for learning because it helped them visualise and represent concepts.

Regarding the spherical coordinates, we institutionalised that, in the Cartesian coordinate system, the location of a point in space is described by an ordered triple, where each coordinate represents a distance. On the other hand, in the cylindrical coordinate system, the location of a point in space is described by two distances (ryz) and an angle (θ). In the spherical coordinate system, we again used an ordered triple to describe the location of a point in space, except that this time, the ordered triple described a distance and two angles. This type of spherical coordinates allowed for the description of a sphere and the cylindrical coordinates of a cylinder. For this, it was formalised according to Stewart (2012, p. 1005):

> Definition: The spherical coordinate system (ρ, θ, φ) of a point P in space (Figure 13), where $\rho = |0P|$ is the distance from the origin to P; θ is the same angle as in cylindrical coordinates, and φ is the angle between the positive axis z and the line segment $0P$. With $\rho \geq 0$, and $0 \leq \varphi \leq \pi$. We must note that this spherical coordinate system is useful in problems with symmetry around a point, and the origin is located at that point.

Figure 13

Spherical coordinates of point P (own adaptation).



Due to the multiple difficulties presented by the students in handling such concepts, we could only work on the conversion between spherical, cylindrical, and rectangular coordinates. As a result, two of the twelve chosen situations could not be resolved because they involved handling those concepts.

Although the formulas to convert spherical coordinates into rectangular coordinates seemed complex, they are typical applications of trigonometry that, for this group of students, are worth reinforcing as an extra activity in class.

CONCLUSIONS

We found that some students do not think trigonometry is a valuable mathematical tool in their professional work. Yet, paradoxically, when asked whether they had ever used *sine and cosine functions,* to name a few*,* they gave answers such as E10's: "*Yes*, *we do use these formulas."* [unit of analysis 421].

In the study, we realised that students progressed in conceptualising elements of the process to be followed to solve a problem situation, gaining familiarity with the properties of the operations to be used, and understanding the contexts in which the integral was presented as an operator and as a function. They also understood trigonometric integrals and improved their ability to relate them to elements of trigonometry. The construction of the integral in an operational sense was crucial in building and understanding the relationships between the AMT and engineering, mathematical symbols, and associated mental objects. Students also advanced regarding the proper use of an essential symbolic system to generalise, formalise, and argue. The students reached an operational level when they could propose adequate solutions to situations they did not know beforehand. This required generalisations, which increased their abstraction capacity, as one must understand not only the operations but their structure, which are elements that help students develop the ability to relate integration in different mathematical problem situations typical of an engineer's work.

Regarding the mathematical knowledge used and the application of mathematical concepts, we observed that students learned how to integrate a variety of products of trigonometric functions that are commonly known as *trigonometric integrals,* some of them supported by specialised mathematical software, however without understanding the technique employed for the integration by trigonometric substitution. Very few students identify that this technique allows converting algebraic expressions one may not be able to integrate into simpler expressions that involve trigonometric functions that allow the integral to be calculated through the techniques described. We emphasise the importance of using specialised software without neglecting the construction of concepts so that the students reach meanings and senses of said constructions to apply them later.

Regarding solving problem situations with trigonometric integrals, we found limited use of relational thinking, referenced in the scarce appropriation of concepts and their possible applications. However, some students could establish emergent relationships between integrals where there were products of powers of trigonometric functions, especially when they identified even powers of sine and odd powers of cosine, and had to decompose the odd power of a cosine. A similar situation happened when they identified the odd power of the sine and even power of the cosine and had to decompose the odd power of the sine using the Pythagorean trigonometric identities. Nevertheless, this did not happen when situations involving the odd power of the tangent and the even power of the secant were presented. Here, the difficulty was in remembering the correct Pythagorean identity and carrying out adequate algebraic processes to apply the respective substitution that would allow visualising the integral in an easier way to calculate. The above allows us to infer that students maintain such a deep-rooted idealisation of the mechanistic processes that it prevents them from identifying that establishing relationships is essential in higher mathematics because they enhance the development of understanding and knowledge of mathematics (Hiebert & Carpenter, 1992).

With the advancement of the investigation, we observed that the students progressively developed relational thinking as a tool that allowed them to identify what type of mathematical knowledge they used and how they applied those mathematical concepts. They could then face various calculus situations where they transformed algebraic expressions or which arithmetic, algebraic, and geometric expressions were related, which entails using flexible strategies. In terms of Carpenter et al. (2005), thinking in this way requires that the students "*look*"(consider) the whole situation to identify a significant number of relationships before starting to calculate and become aware, at least implicitly, of properties and relationships.

The students used relational thinking to simplify calculations, build and learn concepts, and extend procedures to problem situations typical of their engineering formation, which helped us understand that they give a new meaning to integration as an operator that calculates and as a tool when they identify the integral function as $F(x) = \int_a^x f(t)dt$, dependent on the upper limit of the integration. Work focused on the previously acquired knowledge with the understanding of the structure of the proposed situations and the relationships that underlie them. We observe the acquisition of implicit knowledge of algorithms, properties, rules, and operations. The activities focused on relational thinking made it easier to explicit this procedural knowledge exposed in the answers offered. In short, it favoured significant mathematics learning, particularly relating trigonometry as one more element of the AMT, as an organised and systematic system that becomes a tool to deal with everyday problem situations.

The use of technology in mathematics education allows for explaining and visualising definitions, concepts, theorems, sketching graphs, etc., which, at the time, students had difficulty understanding. GeoGebra is applicable when conventional and unconventional surfaces must be sketched, especially when the use of parametric surfaces based on spherical coordinates is required. However, the use of this type and instruments cannot replace the constructions, demonstrations, and formalisations that the teacher must execute with the students for them to achieve meaning from those mathematical entities that they learn.

DATA AVAILABILITY STATEMENT

The data supporting this study will be made available by the corresponding author (EMN), upon reasonable request.

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