


The Construction of Definitions Through Padovan's Combinatorial Model: An Investigation With Didactic Engineering in an Initial Education Course for Mathematics Teachers

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ABSTRACT

Background: Given the oversight of specific topics in the history of mathematics books, this research was motivated by the extensive coverage of the Fibonacci sequence. In addition, the existence of the Padovan sequence, which is considered a Fibonacci cousin, stands out. **Objectives:** Investigating the Padovan sequence, building its definition through a combinatorial model using manipulative materials based on the theory of didactical situations in the initial mathematics teacher education course context. **Design:** The research methodological design follows didactic engineering, and a didactic teaching situation is created to investigate the Padovan sequence. Thus, manipulative material was developed to facilitate the teaching process and the construction of the sequence definition. **Settings and participants:** The interventions were done in the mathematics degree course of a higher education institution in Fortaleza. Five students enrolled in the component of History of Mathematics participated in the study. **Data collection and analysis:** Data were collected during classes, recorded through photos and audio recordings, based on didactic engineering and the theory of didactical situations. **Results:** The most relevant result of this study is constructing the definition of the Padovan sequence through the manipulative material. **Conclusions:** We concluded that the research allowed an investigation of the Padovan sequence, enabling the visualisation of its terms and their integration with other mathematical contents, contributing to teaching these numbers.

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Keywords: Construction of definitions; Didactic engineering; Padovan combinatorial model; Didactic sequence; Theory of didactical situations.

A construção de definições por meio do modelo combinatório de Padovan: uma investigação com a engenharia didática num curso de formação inicial de professores de matemática

RESUMO

Contexto: Diante de conteúdos negligenciados em livros de História da Matemática, surge a motivação desta pesquisa, observando a sequência de Fibonacci sendo abordada de grande forma. Além disso, destaca-se a existência da sequência de Padovan, sendo considerada prima de Fibonacci. **Objetivos:** Realizar uma investigação da sequência de Padovan, construindo sua definição por meio de um modelo combinatório utilizando material manipulável, com base na Teoria das Situações Didáticas, no contexto do curso de formação inicial de professores de Matemática. **Design:** O desenho metodológico segue com a Engenharia Didática como metodologia de pesquisa, sendo elaborada uma situação didática de ensino, como forma de investigar a sequência de Padovan. Assim, foi desenvolvido um material manipulável que facilita o processo de ensino e a construção da definição da sequência. **Ambiente e participantes:** As intervenções foram realizadas no curso de Licenciatura em Matemática de uma Instituição de Ensino Superior na cidade de Fortaleza. Participaram do estudo um total de 5 sujeitos matriculados na disciplina de História da Matemática. **Coleta e análise de dados:** Os dados foram coletados durante as aulas, registrados por meio de fotografias e gravações de áudio, utilizando como embasamento a Engenharia Didática e a Teoria das Situações Didáticas. **Resultados:** Como resultado mais relevante deste estudo, tem-se a construção da definição da sequência de Padovan por meio do material manipulável. **Conclusões:** Pode-se concluir que a pesquisa permitiu uma investigação da sequência de Padovan, possibilitando a visualização de seus termos e sua integração com outros conteúdos matemáticos, contribuindo para o ensino desses números.

Palavras-chave: Construção de definições; Engenharia didática; Modelo combinatório de Padovan; Sequência didática; Teoria das situações didáticas.

INTRODUCTION

History of mathematics authors predominantly debate curiosities about the Fibonacci sequence, neglecting other sequences and other mathematical and historical aspects (Burton, 2007).

Thus, we consider of paramount importance understanding the historical traces that motivated the epistemological and evolutionary process of other recurrent numerical sequences. On the other hand, articles in pure

mathematics explore combinatorial approaches to sequences, mainly the Fibonacci sequence and its contemporaneity.

Briefly, the Fibonacci sequence, created by Leonardo Pisano (1170-1217), is a recurrent numerical sequence of the second order and presents numerous relationships with the golden number (1.61) (Gullberg, 1997). Another sequence is considered a relative of the Fibonacci numbers because it is related to the plastic number (1.32). Notably, the plastic and golden numbers are the only two solutions of a numerical set called morphic numbers (Ferreira, 2015; Alves & Catarino, 2022).

This Fibonacci-related sequence is known as the Padovan sequence and was created by Richard Padovan (1935). Marohnic, Kovacic, and Radisic (2013) claim that Padovan's study was based on architect Hans van der Laan's (1904-1991) work, who discovered a new irrational number, the plastic number. However, we must note that, according to research, this number was previously studied by Gérard Cordonnier (1907-1977); therefore, this sequence is also known as the Hans van der Laan or Cordonnier sequence (Alves & Catarino, 2022).

Faced with this relationship between these numbers, the motivation to explore other sequences arose, enhancing their respective epistemological and mathematical developments. In fact, the unpublished nature of Padovan's mathematical evolution focused on teaching and constructing definitions is verified, providing the subjects participating in the research with a unique opportunity for study. This fact justifies the interest in developing a teaching proposal for the study of Padovan sequences, emphasising the combinatorial model of these numbers.

Given the above, we believe building teaching scenarios connected with students' needs is necessary. Thus, one can see the challenge imposed on the mathematics teacher to make classes more attractive, arousing students' interest in constructing mathematical concepts through didactic situations that stimulate curiosity. In this sense, we seek, among the theories of didactics of mathematics of French origin, the one that analyses the obstacles during the teaching process and studies the didactic and cognitive aspects, finding the theory of didactical situations (Brousseau, 1986). The selection of the research methodology also followed the same French bias, attributing this work to didactic engineering (Artigue, 1988), resulting from Brousseau's study and the practices of didactic situations.

Research on the didactics of mathematics (*didactiques de mathématique*) shows that the first studies were developed in France from the 1960s through the 1980s at the founding of the International Commission on Mathematical Instruction, reporting a strong interest in the classic trinomial: teacher-student-knowledge. In Brazil, known as mathematics education, we can say that when carrying out a didactic transposition, the teacher is the main element of the trinomial. According to Alves (2018), two elements deserve to be highlighted with the arrival of these didactics in Brazil: the first, regarding the movement of the constitution and scientific identity of research, concerning the teaching of science and mathematics; and the second, related to the effective field of action, where the student becomes a subject analysed very clearly by the teacher.

The didactics of mathematics is defined as:

One of the trends in the broad area of mathematics education, whose object of study is the elaboration of concepts and theories that are compatible with the educational specificity of mathematical school knowledge, seeking to maintain strong links with the formation of mathematical concepts, both at an experimental level of pedagogical practice and in the theoretical territory of academic research. (Pais, 2011, p. 11)

Although the focus is on the teacher, we should emphasise students' importance, as they are also placed at the centre of the learning process, and the teacher is responsible for guiding them correctly towards specific problem situations to be resolved. These didactics of mathematics gather investigative research on the emergence of possible obstacles in the epistemological construction of mathematical concepts in teaching situations. Given this scenario, this work focuses on the construction of a definition, presenting an investigation of the Padovan sequence to transform it into content to teach.

Didactic engineering structures the knowledge used in the activity, initially raising hypotheses and analysing them later. This methodology allows for didactic innovation, which is the organisation of methodological research procedures being developed in the classroom. Thus, we can overcome obstacles in any branch of mathematics by proposing different didactic situations to foster opportunities for the construction of students' knowledge.

Building on the above, we intend to (re)think teaching methodologies educators and researchers employ when addressing the Padovan sequence in the History of Mathematics component for pre-service teacher education

courses. Therefore, we observe the contribution of French didactics, especially didactics of mathematics, relating them to studies on mathematical objects and their relationships experienced in the classroom.

Based on Vieira, Alves, and Catarino (2022), Spivey (2019) and Spreafico (2014) on the combinatorial model of the Padovan sequence, Fibonacci and the board notion, we launch our guiding question: How to construct the definition of the Padovan sequence based on his combinatorial model in a pre-service math teacher education course? From this guiding question, the general objective of this research is to investigate the Padovan sequence, building its definition considering its combinatorial model via manipulative resources, based on the theory of didactical situations, in a pre-service math teacher education course.

The first step is to pose a problem based on Brousseau's theory of didactical situations (1986), focusing on constructing the definition of Padovan's sequence to support the initial math teacher education.

THEORETICAL FRAMEWORK

Without a doubt, mathematical teaching and learning are not trivial for many students, requiring alternatives to stimulate their reasoning and interest. Thus, in the mid-1980s, a research methodology emerged in France to improve existing practices in the education system, also seeking an understanding of the events of mathematical learning.

With this, the methodology of didactic engineering emerges. Artigue (1988) says that his work is similar to an engineer's, relying on technical, scientific knowledge within their domain and being compelled to use more complex objects than those depurated from science. Furthermore, this methodology enables the study of existing practices in the classroom, provides resources for teacher education and analyses their respective activity and didactic transposition of mathematical content (Chevallard, 1991).

Given the origin of the perspectives of analysing the teacher's role, around the didactics of mathematics, there is the teaching of Padovan's combinatorial model, emphasising the initial education of mathematics teachers. Therefore, this research used didactic engineering, presented as microengineering or macroengineering, where the first has a more restricted look at classroom practices, while the second has a more global look. Therefore, we use microengineering in this work to teach the object of the mathematical

study. Artigue (1995) also says that the use of this research methodology is somewhat complex since the data collected in the classroom “are not easy to develop in practice” (Artigue, 1995, p. 36).

Didactic engineering allows the study of existing phenomena in the classroom, providing resources for teacher education. In this way, it is possible to analyse the relevance of the teacher’s role and their action of transposing the content didactically around scientific knowledge. Research in the French area of the didactics of mathematics enhances the conditions and scientific experiments in the classroom. Thus, we have the approach of the combinatorial study of the Padovan sequence with the methodology of didactic engineering in addition to the theory of the didactical situations for pre-service math teacher education courses with an emphasis on the training and learning of the mathematics teacher, as well as the evolutionary process of the Padovan sequence.

This work is divided into four phases: preliminary analysis, design and a priori analysis, experimentation and a posteriori analysis and validation. In the first phase, we identify problems relating to teaching and learning, searching the literature for works and books on the object of mathematical study and listing the epistemological, cognitive, and didactical elements (Artigue, 1995).

In the design and a priori analysis, we select the variables (microdidactics or macrodidactics, discussed later) and develop teaching situations based on the epistemic-mathematical field, seeking to achieve the research objective. Almouloud (2007) states that:

A priori analysis is extremely important, as the success of the problem situation depends on its quality; furthermore, it allows the teacher to control the performance of students’ activities and identify and understand the facts observed. Thus, the conjectures that will appear can be considered, and some can be the subject of a scientific debate. (Almouloud, 2007, p.176)

The teaching situations developed in the previous phase are applied in experimentation, and the collected data must be registered (Alves, 2016).

Lopes, Palma, and Sá (2018) state that,

Initially, it consists of the period of application and experimentation of previously planned activities, collecting data on the investigation. Secondly, it refers to the analysis of

the results that will be obtained in the investigation. This phase is based on the analysis of all data obtained in experimentation during teaching sessions, as well as productions inside or outside the classroom. (Lopes, Palma, & Sá, 2018, p. 164)

Furthermore, a teaching contract must present the due responsibilities of teachers and research participants. However, the didactic contract is sometimes breached when students are not interested in the learning process.

Finally, in the last phase, a posteriori analysis and validation are done, analysing the data collected in the previous phase and comparing them with the a priori analysis, thus validating the formulated hypotheses. Validation can be internal, analysing only the students participating in the research, or external, comparing participants who used the research methodology with subjects who did not (Laborde, 1997).

Almouloud (2007) argues that adjustments and corrections may be required during the application. Thus, we move on to the last phase, evaluating the results so that didactic knowledge can contribute to content transmission. Therefore, in the experimentation phase, the elements are validated by comparing the results discussed and analysing the likely evolution of the engineering.

Aiming to support the phases of didactic engineering, we thought it convenient to use a teaching methodology that provides students with a place to learn and exchange information, the theory of didactical situations, based on teaching situations.

This teaching methodology encourages students to solve teaching situations, promoting an investigation during the mathematics teaching and learning process (Brousseau, 1986). It is noteworthy that the didactic situation, according to Brousseau (2006), is “the model of interaction of a subject with a specific environment that determines some knowledge” (Brousseau, 2006, p. 19). So, one must not forget the presence of the *milieu*, which is the context where the didactical situation is applied. In fact, the theory of didactical situations is divided into four situations: action, formulation, validation, and institutionalisation.

According to Alves (2016), in action, the subject has the first contact with the posed problem situation, an activity composed of questions with direct and objective statements. Thus, participants will try to solve it by searching through the acquired knowledge. Brousseau (2002) states that:

The sequence of “action situations” constitutes the process by which students form strategies, that is, “teach themselves” a method to solve their problems. This succession of interactions between the student and the environment constitutes the “dialectic of action”. We use the word “dialectic” and not the word “interaction” because, on the one hand, the student can anticipate the results of their choices and, on the other, their strategies are somewhat propositions confirmed or invalidated by experimentation in a kind of dialogue with the situation. (Brousseau, 2002, p. 9, our translation)

Participants transform ideas into more technical and formal languages in the formulation, aiming to conjecture theorems and properties (Vieira, Alves, & Catarino, 2019). The subject must evolve in each phase (situations) of the theory, allowing information from previous phases to be applied and the idea that reasoning is an experiment.

The validation situation occurs with the validation of the resolutions presented in the action phase, proposing discussions. Oliveira and Alves (2019) state that at this point, students must have already internalised the mathematical concepts, enabling the use of methods to demonstrate the verified mathematical concepts.

Finally, in institutionalisation, the teacher analyses the resolutions presented, revealing the objective of the problem situation (Alves, 2019). Vieira, Alves, and Catarino (2020) affirm that,

In this last stage, the teacher assumes the situation again, identifying and recognising the knowledge built in the other stages discussed. The resolutions are checked, and the real intention of the proposed activity is then revealed, evaluating the transfer of knowledge (*connaissances*) and scientific knowledge. (*savoir scientifique*)

Then, the first phase of didactic engineering begins with surveying the theoretical framework around the object of study and establishing the boundaries of the respective epistemic-mathematical field. Furthermore, it is pertinent to discuss the teaching contract determining the responsibility of each teacher and student, respecting the rules agreed upon during the teaching and learning processes. Pais (2011) defines a didactic contract as “an appropriate notion to understand the educational phenomenon at the more specific level of

the classroom, although, in the reality of everyday school life, unforeseeable events occur, making it difficult to achieve the proposed objectives”.

It is worth noting that the teaching contract does not always work because sometimes the student is not interested in solving the problem situations posed, which results in a breach of the contract. Therefore, in the experimentation phase, we collect data from students in the final stage of the didactic engineering to validate it or not.

A crucial characteristic of the theory of didactical situations is its relevance in relation to mathematics and its epistemological study, being expressed in different ways, highlighting the notion of epistemological obstacle. Epistemological obstacles are forms of knowledge considered relevant and successful in particular contexts beyond school contexts, which became false or inadequate at a certain point (Brousseau, 1986).

These studies were based on Bachelard, who described a list of obstacles of an epistemological nature. The idea of an obstacle is formed and modified as the teaching process takes place, expanding beyond epistemology to didactics, psychology, and others.

Mathematical definitions indicate the advancement of knowledge, with the epistemological study of definitions facilitating the identification of historical concepts. Lakatos (1980) considers several elements, such as conjecture, proof of the conjecture, and the definitions under construction, the hidden lemmas that allow the emergence of new concepts and other proofs.

In fact, this research deals with mathematical concepts and definitions under construction in mathematical research, involving the content of numerical sequences and combinatorics, allowing these conditions to have a similar conceptual baggage when faced with a situation involving a “new” concept. The situation used in this investigation was one of classification, demonstrated through the constructed definition. Investment was also made in an exploratory manner at the primary and secondary levels, working in a classificatory situation on the concepts of combinatorial interpretation of the Fibonacci sequence through the concrete material developed for this purpose. The elaboration of the situation, the analysis and validation of these situations are possible given the theory of didactical situations.

The characterisation of the definition activity is carried out by representation through the developed manipulative material, describing the functioning of mathematics, using Balacheff’s model (1995), based on Brousseau’s theory of didactical situations (1986).

The study of definitions addresses the concepts by Aristotle, which develops logical and linguistic components, studying discourse; Popper, proposing a theoretical study of definitions; and Lakatos, paying attention to the construction of the concepts covered in the definitions, thus carrying out a heuristic study.

Lakatos' (1980) conception also addresses Aristotle's (1965) and Popper's (1985) conceptions, focusing on the process of generating concepts, reserving a privileged place for definitions. The definition of the problem from Lakatos' (1980) perspective primarily presents the situation of classification, delimiting the concept of the Padovan sequence.

The representation system in Lakatos (1980) can be done through games. Therefore, we have the development of manipulative resources involving the Fibonacci combinatorial interpretation to allow us to handle the pieces and obtain the Padovan combinatorial interpretation

Aristotle's (1965) conception is carried out through the process of definition by gender and specific differences. The classification of the problem is more general, presenting the problem and its delimitation. The controls are well defined, verifying the unity of the concept, as established by Aristotle (1965). Therefore, redundancies must be prohibited, questioning the existence of concepts.

Popper's (1985) conception is distinguished by its rejection of Aristotle's essentialism. Popper's main contribution to the construction of definitions lies in operators focused on the construction of a scientific theory and control structures. Problem selection is based on competing theories.

The concepts of discrete mathematics involved in combinatorial and arithmetic problems verify these different conditions and serve a particular definition activity, as in Ouvrier-Buffet (2003). Thus, we have the proposal for this activity, with the construction of the definition of the recurrence of the Padovan sequence, considering its originality in relation to the mathematical object involved and the elaboration of the manipulative material.

METHODOLOGY

This section describes the methodological procedure in this research. Initially, we surveyed the theoretical framework, covering the research methodology of didactic engineering, the teaching theory of the theory of didactical situations and the mathematical object (Padovan sequence and its

combinatorial approach), thus conducting the preliminary analyses of the didactic engineering. Regarding the mathematical object, works in the pure mathematics field were discussed on the Padovan sequence to recover the state of the art of this sequence and allow didactic transposition in the classroom in the following phases.

To apply the research in an initial teacher education course, we selected the History of Mathematics component of the mathematics degree course. The target audience was because the curriculum component has historical development and representations of numbers in its syllabus, enabling the study of linear recurring sequences. In fact, the presence of the content of linear recurring sequences was noted when discussing the history of mathematics. Furthermore, students must have some prior knowledge relevant to the study of the mathematical object in question, such as that acquired in the Linear Algebra component, according to the course syllabus (Education, 2023).

Therefore, the research was applied to a sample of five students enrolled in the component above, allowing an investigation based on didactic engineering and the theory of didactical situations. Indeed, the research has an unprecedented character, allowing the contribution of the manipulative material as a way of transposing the mathematical content to the teaching area, in addition to envisioning the visualisation of the terms of the sequence before the construction of the definition.

Then, we gave expository classes to students of the initial teacher education course, addressing recurrent numerical sequences, the Fibonacci sequence, and the notion of the board. Four classes provided participants with a mathematical basis for constructing the definition of the new sequence investigated.

Subsequently, we developed a problem situation and analysed possible students' behaviours during the application based on the theory of didactical situations. At that moment, we observed the second phase of didactic engineering.

Going on with the methodological procedure, the proposed activity was tried in the initial mathematics teacher education course, continuing the third phase of the didactic engineering. During the experiment, the participants were analysed based on the theory of didactical situations, and the data was recorded and stored for later discussion in the research. Finally, the predicted data was compared with the applied and validated data, according to the internal

validation of the didactic engineering, concluding the fourth and final phase of the process.

PRELIMINARY ANALYSIS

In this section, we carried out a bibliographic survey, referring to the Fibonacci combinatorial model and the notion of the board to begin defining the “new” sequence. After that, we listed the works in the pure mathematics and history of mathematics fields to explore the genesis of this sequence, highlighting relevant points such as the notion of the board, properties, and other mathematical concepts. Thus, elements of an epistemological order are listed so that they can then be transformed into content to teach.

Initially, we study the recurring numerical sequences, evidenced by the Fibonacci sequence, addressing its historical aspect and disregarding other important aspects of the mathematical contribution (Burton, 2007). We can say that a recurring numerical sequence is formed by an infinite list of terms. To obtain these terms, one must calculate it using a recurrence formula. Furthermore, some initial terms are defined depending on the order of the sequence. An example of this is the Fibonacci sequence, as a second-order sequence.

These numbers have their genesis in the problem of pairs of immortal rabbits, generating the terms of the Fibonacci sequence (Gullberg, 1997). This sequence was developed by Leonardo Pisano (1170-1217), presenting the terms 0, 1, 1, 2, 3, 5, ... Thus, its recurrence formula is given by: $F_n = F_{n-1} + F_{n-2}, n \geq 2$ and the initial terms $F_0 = 0, F_1 = 1$.

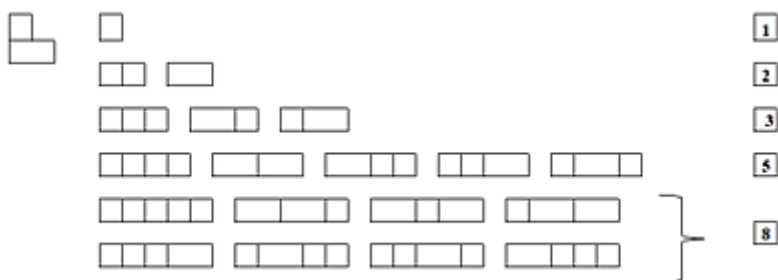
Therefore, several applications and approaches involving this sequence are neglected in the history of mathematics books. Based on this, this research will address the combinatorial interpretation of these numbers, allowing a way to visualise the terms of this sequence.

In view of this, it is important to mention the notion of the board, established by Spreafico (2014), which deals with a set of squares called cells, enumerated, describing a specific position. From this, we deal with the Fibonacci combinatorial model, in which f_n represents the number of forms of tiling on the $1 \times n$ board, using 1×1 squares and 1×2 dominoes. In this way, we have $f_n = F_{n+1}$. (Spivey, 2019). Figure 1 shows the initial Fibonacci tilings,

from $n=1$ to $n=5$, observing on the right side the correspondence of the Fibonacci number to each analysed board.

Figure 1

Fibonacci combinatorial model. (Adapted from Spivey, 2019).



A PRIORI ANALYSIS AND DESIGN OF DIDACTICAL SITUATIONS

In this section, we develop the problem situation, analysed based on the microdidactic variable, since the proposed question refers to the organisation of an experiment, as foreseen in didactic engineering (Almouloud, 2007).

Given this, we examined the preliminary analyses, addressing the epistemic-mathematical field of research, thus exploring the combinatorial interpretation of the Fibonacci sequence. With this, an epistemological, cognitive, and didactic analysis was possible. Aiming to facilitate the teaching and learning process, we created the manipulative resource containing pieces for constructing Fibonacci boards and the “new” sequence under construction.

Next, we discussed the didactical situation composed of problem situations based on the theory of didactical situations, allowing an analysis of the methodology and concepts addressed. During the posing of the problem situation, the didactic variables relating to the contents developed in the epistemic-mathematical field are analysed: notion of a board (Spreafico, 2014), Fibonacci combinatorial model (Spivey, 2019), construction of definitions, Padovan sequence, and Padovan combinatorial model.

Problem situation: Generally speaking, we can observe contemporary research on the study of recurring numerical sequences and their countless

generalisations, usually neglected in the history of mathematics books (Burton, 2007). An interesting approach is the combinatorial interpretation of the Fibonacci sequence via tiling. It is worth mentioning that a board comprises squares called houses, cells, or positions. These positions are numbered, and these numbers describe the position. A given board will be called an n -board (Spreafico, 2014). Thus, based on the recurrence of the Fibonacci sequence, given by $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, the initial values are given by $F_0 = 0, F_1 = 1$. This way, the manipulative resources and available parts can be used to construct Fibonacci tiling. To do this, consider the square (size 1×1) and the domino (size 1×2), both white, as available pieces. Consider f_n the number of configurations to tile a board of length n with squares and dominoes. That said, the number of ways to tile these pieces for a board of size 1×1 is one; for a 1×2 board, there are two ways; for a 1×3 board, there are three ways; for a 1×4 board, there are five ways; and for a board of size 1×5 , there are eight ways. With this, we have that the number of ways to cover an n -board with 1×1 squares and 1×2 dominoes equals $f_n = F_{n+1}$. Figure 1 presents the Fibonacci tilings.

Given this information, new pieces are available: a gray trimino, defined as a rectangle of size 1×3 ; a black square 1×1 and the blue domino 1×2 ; any tile is formed by the arrangement of defined shapes. The black square is intended to complement the empty tiles, subject to the rule of being inserted only at the beginning and only once on each tile.

Thus, how many shapes of tiles can be obtained for a board of size 1, 2, 3, 4, and 5, knowing that for $n=0$, we have a value equal to 1? Do these numbers form any numerical sequence?

For the proposed activity, students should discuss the following: Definition 1. Numerical sequence; Definition 2. Fibonacci sequence; Definition 3. Notion of the board; Definition 4. Fibonacci combinatorial model; Definition 5. Rules for configuring the pieces of Padovan's combinatorial interpretation; Definition 6. Padovan sequence; Definition 7. Padovan combinatorial model.

Action phase: students must observe the Fibonacci tiles to understand their construction. Thus, we seek to appropriate the manipulative material to understand the combinatorial interpretation of the Fibonacci sequence with the established rules and pieces. Students should then investigate the construction of Definitions 1, 2, 3, and 4. Faced with this problem situation, students' possible estimated errors will be in relation to the rules and configurations of

the pieces, highlighting the insertion of the black square. For this obstacle to be resolved, students must understand that the black square can only be inserted once and at the beginning of the tile, noting that it may not appear in the other available tiles.

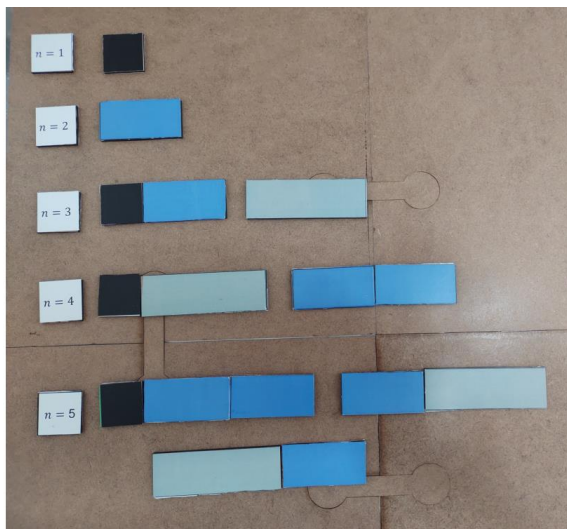
Formulation phase: The aim is for subjects to manipulate the pieces available, following the defined rules, to assemble the boards of sizes 1×1 , 1×2 , 1×3 , 1×4 and 1×5 ($n = 1, 2, 3, 4$ and 5). For the 1×1 board, obtaining only one tile shape is possible with the presence of the black square. For the board of size 1×2 , it is possible to obtain only one way of tiling, with the presence of the blue domino. For the 1×3 board, it is possible to obtain two tiling shapes: a black square concatenated with the blue domino and a grey trimino. For the 1×4 board, it is possible to obtain two tiling shapes: a black square concatenated with the grey trimino and two blue dominoes concatenated. Finally, for the 1×5 board, it is possible to obtain three tiling shapes: a black square concatenated with two blue dominoes; a grey trimino concatenated with two blue dominoes, and two blue dominoes concatenated with a grey trimino. Figure 2 represents the construction of these cited cases, obtaining the terms arising from a possible sequence.

Thus, the presence of the following numbers belonging to the numerical sequence is observed: 1, 1, 1, 2, 2, and 3. It is noteworthy that the first term, $n = 0$, was defined in the activity statement. In this phase, students address Definition 5 and investigate Definition 4 so that they can construct Definitions 6 and 7. The possible errors foreseen are in relation to the construction of the tiles, paying attention to the rules and numbers that appeared according to each value of n . Therefore, students must observe the maximum number of tiles possible, and cannot ignore and/or forget any case, given this combinatorial interpretation. It is essential to mind another possible error, as students may get confused and think that the tiles occur by obtaining the early term of the sequence, just like for Fibonacci.

Validation phase: in this phase, students must observe the terms obtained and thus identify the recurrence rule for this numerical sequence. Hence, it is possible to determine that the next term (sixth term) will have value 4. This value comes from the sum of the third term and the fourth term, ignoring the fifth term. Thus, recurrence can be established for the studied sequence (S_n) as $S_n = S_{n-2} + S_{n-3}$, requiring the initial values $S_0 = S_1 = S_2 = 1$ for $n \geq 3$.

Figure 2

Combinatorial interpretation of the “new” sequence under study.



It is important to note that until now, the teacher must not interfere in the activity, not presenting the sequence or naming it. With this, students can assign any name to the investigated sequence. Then, subjects can validate Definition 7. In this phase, possible errors are predicted, referring to the recurrence of the studied sequence, without visualising the jump between the immediate previous term. To do this, students must observe the terms in the sequence to obtain the next term, according to the numbers they got when constructing the tiles.

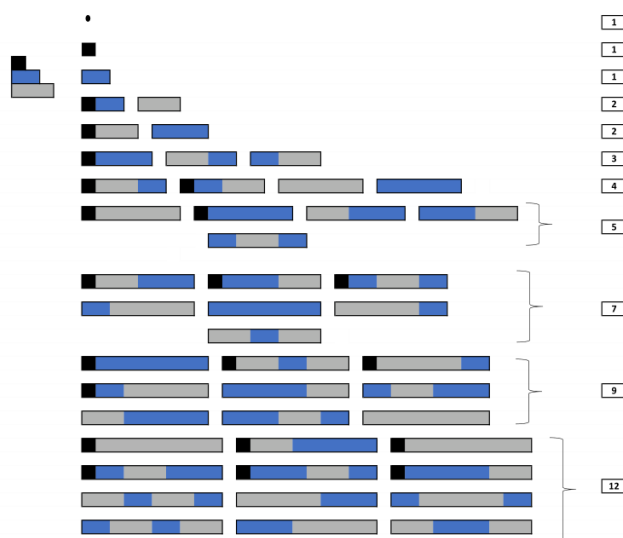
Institutionalisation phase: After the discussions, the teacher must resume the position of the activity, checking students' concerns. At this moment, there must be a discussion around the students' arguments, also identifying the erroneous reasonings that occurred. The institutionalisation of the Padovan sequence and its combinatorial interpretation for the cases mentioned above must occur ($n = 1, 2, 3, 4$ and 5). With this, the teacher must present the Padovan sequence, naming the numerical sequence studied and investigated by the students through the manipulative material and its combinatorial form. In fact, the definition of the Padovan sequence is constructed, enabling students to obtain its terms through the combinatorial

approach. Furthermore, the teacher must demonstrate the theorem relating to Padovan's combinatorial interpretation, allowing a visualisation of this combinatorial approach.

Thus, the Padovan sequence is a third-order numerical sequence, first discovered in 1924 by the French architecture student Gérard Cordonnier. Independently remodelled by the French monk-architect Dom Hans Van der Laan (1904-1991), the Padovan numbers are known to have a historical association with Cordonnier, Padovan, Van Der Laan, and Stewart, also known as the Cordonnier sequence. In his book, Richard Padovan attributed the sequence to Van der Laan (Padovan, 2002). Its recurrence is given by the relation: $P_n = P_{n-2} + P_{n-3}$, where $P_0 = P_1 = P_2 = 1$ for $n \in \mathbb{N}$.

Figure 3

Combinatorial model of the Padovan sequence. (Vieira, Alves & Catarino, 2022).



Concerning the Padovan tiling, we have that for $n \geq 0$ the possible tiling of a $1 \times n$ board, with black squared, blue domino, and grey trimino tiles is given by: $p_n = P_n$, being p_n the number of ways to fill the $1 \times n$ board and

P_n the n th term of the Padovan sequence (Vieira, Alves, & Catarino, 2022) (see Figure 3).

Students may not perceive the study of zero definitions as typifying provisional notions but rather as ideas that arise during the resolution of the proposed activity. We observed that such a synthesis also allows for establishing students' conceptions about the mathematical definition and evaluating evolutions during subsequent experiments involving other mathematical concepts and types of problems.

Such an approach is consistent in building a fundamental situation in mathematics. We can identify students' notions about the mathematical concept and help them understand the proposed mathematical problems.

EXPERIMENTATION

The experimentation phase was done in a higher education institution in Fortaleza, Ceará, for the mathematics teaching degree course. The component observed was History of Mathematics, which was mandatory and had five students enrolled. The application took two weeks, with four classes lasting 2 hours each.

The classes were based on the theory of didactical situations, with a didactic teaching situation created using a list of exercises and a manipulative material. To this end, the participants held discussions to develop resolution strategies described in the table and material (game).

The didactic contract is settled, and the teacher and students agree on mutual expectations. Thus, the relationships of knowledge and how both implement and treat it are included.

This stage is also characterised by applying the whole structure already organised until then, observing learning situations and involving the concepts foreseen in didactic research, not following the dynamics of a traditional or standard class (Pais, 2011). Therefore, during this period, the organisation of the class must be based on the theory of didactical situations and focused on collecting students' data so that teaching-learning can be analysed.

During the experiment, the teacher must put into practice the didactic sequence elaborated in the conception and a priori analysis, observing the moment of resolution of the activities proposed by the students, to be analysed later in the next phase.

Initially, a study of recurring numerical sequences presented the Fibonacci sequence and its combinatorial approach via tiling. Thus, a discussion around the notion of a board was carried out, allowing the subjects to obtain a basis for the following investigation.

To collect data, audio recording procedures and photographic records were carried out during discussions in the classes under analysis. All students signed a free and informed consent form to authorise the recordings.

A POSTERIORI ANALYSIS AND INTERNAL VALIDATION

During application, it is crucial to highlight possible corrections and local adjustments, and only a few elements considered most relevant in this research are indicated during the experimentation process. Above all, after experimentation, we go back to a posteriori analysis to evaluate and analyse the results and explore the data so that there is a contribution of didactic knowledge during the transmission of the content.

After the a posteriori analysis, the elements used in the experimentation phase of the didactic engineering must be validated by comparing the results discussed by the students, who may have raised questions about an analysis of the possible evolution of the suggested engineering. The problem situation discussed must be considered a micro didactic variable which, according to Almouloud (2007), is “relative to the local organisation of the sequence, that is, the organisation of a session or phase of experimentation”.

The application began with distributing the exercise list and rules for handling the manipulative material. The validation took place with the elaboration of a didactic situation through an activity proposed to students, to instigate the participant’s investigative side, proposing the necessary knowledge for solutions and definition construction. Given this, the problem situation allowed understanding of the definition of the recurrence of the Padovan sequence via its combinatorial approach, discussed according to the didactic variable. The validation occurred internally, with no discussion with other applications and environments.

The problem situation aims to construct the definition of the Padovan sequence, rescuing Fibonacci’s combinatorial approach and the notion of a board. Thus, we encouraged a discussion around numerical and recurring

sequences, observing the students' difficulty in establishing the recurrence of specific sequences.

Next, we carried out observations and discussions based on the registers stored during the applications, focusing on didactic engineering and the theory of didactical situations, internally validating this research.

With this, the manipulative material, the rules for building the Fibonacci combinatorial model, and the model were presented. Next, the construction rules and available parts for a new sequence were presented. Figure 4 depicts the action phase of students A, B, and C, trying to build the combinatorial model of the new sequence under study.

According to Brousseau (1986), this moment is when the student presents a solution strategy, which can be written or oral. Thus, one can observe the subjects' idea of assembling Padovan tiles, given the available pieces and the respective tile sizes.

Figure 4

Action phase occurring during experimentation.



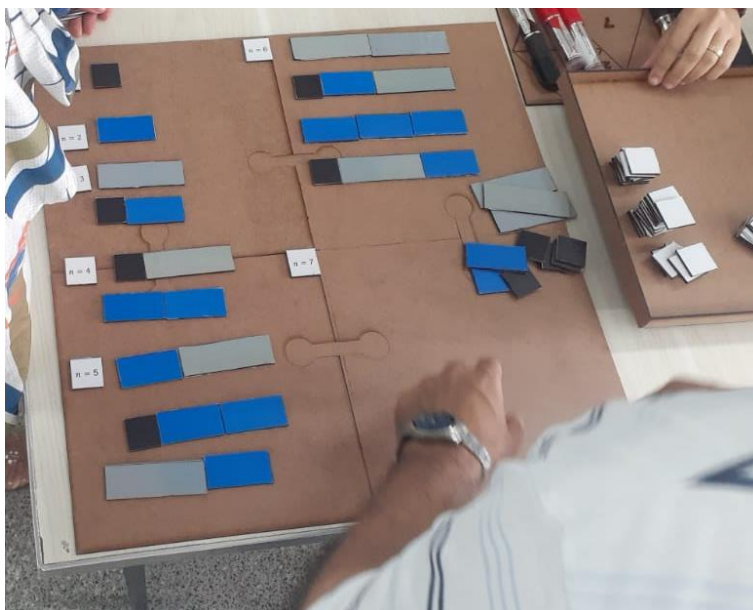
At this stage, the students understood the black square rule, but Student B had not understood the notion of a board, resulting in the construction of disordered tiles with values of n that were different from the size of the final

piece. To assist Student B in understanding it, Student C explained the construction of the $1 \times n$ boards. Therefore, it is clear that students' discussions are valuable because they can help them better understand the content covered. It is also interesting to mention that the subjects knew neither the combinatorial interpretations of the sequences nor the Fibonacci sequence.

During the formulation, students developed beyond the requested proposal. Thus, they constructed the tiles for the values of $n=6$, $n=7$. Initially, when determining the values up to $n=5$, some students could not calculate the terms that followed, i.e., they could not identify the recurrence and construction rule of the terms. Therefore, they chose to develop more tiles. Figure 5 shows the formulation phase by Student B and Student D, continuing the combinatorial model for later n values.

Figure 5

Formulation phase occurring during experimentation.



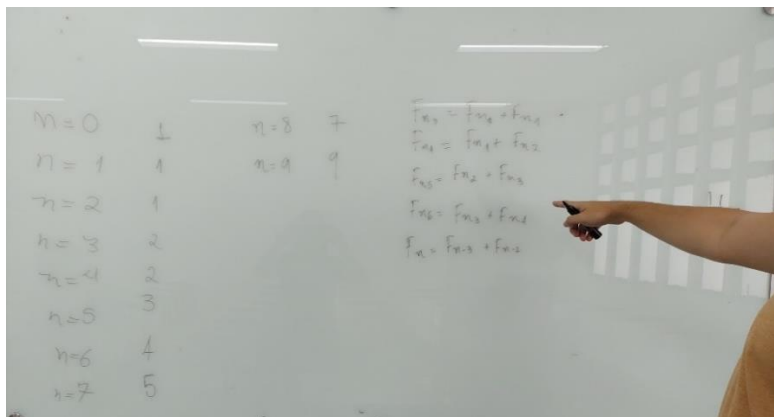
In the validation, Figure 6 shows Student E's construction of the definition of the recurrence of the sequence. We can see that the student

determined more terms of the sequence after building the combinatorial model, making its recurrence effective based on the calculations for the predecessor terms.

We realised how students relied on the numbers obtained by the manipulable material, which allowed them to obtain the recurrence of the new sequence under study. Based on Brousseau (1986), we realised that the participants remained involved in the activity, demonstrating and convincing other colleagues of the arguments used to solve the problem.

Figure 6

Validation phase occurring during experimentation.



During the resolution, we identified epistemological and cognitive obstacles that were overcome through discussion, exchange of information, and contribution from the participating group. This fact is evidenced when one student says, “If Fibonacci is the sum of two terms, this also has to be the case, it remains to be seen what these two terms are”. This statement shows the difficulty in developing the Padovan sequence recurrence formula, with some terms of this sequence being available.

Constructing definitions portrays students’ search for knowledge, taking into account the philosophers cited in the study of this research. Thus, given Popper’s (1985) conception, the representation system relates theories and concepts, as with the sequences studied (Fibonacci and Padovan).

Operators are done based on the construction and testing of local axioms by handling the manipulatives. Controls are based on the formulation of statements and experimental tests, handling the available parts to obtain the terms of the Padovan sequence through tiling. Given the above, we have the development of activity focused on constructing definitions and mobilising analogies and mathematical fields. The defining activity is mainly guided by classifying and categorising objects and their nomenclature. The construction of the definition in question will be guided by seeking the smallest number of initial conditions to obtain the highest number of results. One of the main difficulties of working with mathematical definitions is choosing a mathematical concept and the a priori exploration of this concept to determine the appropriate problems and consider several possible definitions or even conjectures and proofs.

Based on Lakatos (1980), we observed that students interpret the Padovan sequence without knowing its recurrence and terms. The sequence is not named so that students can investigate and define the concept in relation to the combinatorial approach of Padovan numbers through tiling. Indeed, one can highlight the change in representation and language related to a transition from geometry and combinatorics to the algebraic topology of these numbers, which Lakatos (1980) think is insufficient.

Given Aristotle's (1965) conception, the representation system is offered based on rules and parts established by the manipulative mathematical object so that students can handle them and obtain the terms of this sequence under study. Using simple language, the operators are studied demonstrating the equivalence between the Fibonacci and Padovan definitions.

Finally, the activity focused on constructing definitions, mobilising analogies, and mathematical fields. The definition activity is mainly guided by classification, categorisation of objects and their nomenclature.

FINAL CONSIDERATIONS

Following the research methodology of didactic engineering, preliminary analyses began with the investigation of pure mathematics works related to Padovan and Fibonacci sequences. The importance of books, articles, and theses was noted as fundamental sources for advancing research into those numbers, rescuing their historical, cognitive, and didactic aspects.

Based on this, we fostered a study on recurring numerical sequences, taking the Fibonacci sequence as an example. This study developed the

epistemic-mathematical field for discussing the Fibonacci combinatorial model and the notion of a board. The objective was to obtain the definition of the Padovan sequence through its combinatorial model.

Then, we developed a problem situation for initial mathematics teacher education courses, thus transposing the theme into the didactic-cognitive context.

The history and evolution of sequences in general were explored during the discussion of mathematical concepts. Manipulative resources were used as facilitators to construct the definition. Furthermore, the didactic teaching situation was based on the theory of didactical situations, analysing possible student behaviours and comparing the outcomes.

Indeed, the assumptions of the adopted research methodology (didactic engineering) and teaching theory (theory of didactical situations) allow a reflection on the content and its corresponding didactic transposition. It is essential to highlight that using manipulative resources allowed a deeper understanding of the content, offering a unique perspective in relation to the combinatorial approach of sequences in the classroom.

Given didactic microengineering and didactic transposition, it is essential to explain that the sequence was studied by creating a teaching situation based on the theory of didactical situations. This situation was analysed and discussed through a previously selected problem situation to understand the historical-evolutionary process of the Padovan sequence, exploring the construction of definitions and the combinatorial approach.

Manipulatives played a crucial role in facilitating the teaching and learning process. Through games and their combinatorial interpretations, students could visualise the terms of the Fibonacci and Padovan sequences, integrating the content of the sequences with combinatorial analysis.

Finally, this research seeks to expand the investigation and dissemination of the work to professors-researchers, highlighting the interest in education and improvement of mathematics teachers' classroom practices.

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AUTHORSHIP CONTRIBUTION STATEMENT

FRVA and PMMCC carried out project supervision. FRVA, PMMCC and RPMV conceived the idea presented and discussed in this research. RPMV developed the theory, adapted the methodology to the classroom context experienced, created models, and collected data. FRVA, PMMCC, and RPMV analysed and discussed the data to make the final research contribution.

DATA AVAILABILITY STATEMENT

The authors agree that the data supporting the results of this study are available upon reasonable request at the authors' discretion.

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