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The construction of the concept of variable by elementary and secondary school students: contributions from the Theory of Conceptual Fields

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ABSTRACT

Context: This article presents an excerpt from a study that aimed to develop the concept of variable among 8th-grade elementary school students. **Objectives:** The objective is to analyze the strategies and arguments used by 8th-grade elementary school students when confronted with the concept of variable in the study of algebra. **Design:** This is a qualitative educational study with case study characteristics, in which we conducted a teaching intervention that was constructed, experienced, and analyzed based on the Theory of Conceptual Fields. **Setting and Participants:** Seven students from the 8th-grade class of the public school in Duque de Caxias, Rio de Janeiro, where the intervention was conducted. **Data Collection and Analysis:** We filmed the intervention and transcribed the participants' statements. The Theory of Conceptual Fields underpinned the research, offering the notions of concept, conceptual field, schema, representation, operational invariant, algebra, and algebraic thinking. **Results:** The results show that students develop a range of strategies and arguments to deal with the notion of variable. Furthermore, they have difficulty dealing with expressions like "mx + n." Whole numbers and their operations are associated with the construction of the concept of variable, and the diversity of the letter representing the variable can be a hindrance when not well understood by students. **Conclusions:** Understanding algebraic expressions involves distinguishing between the concepts of unknown and variable; therefore, these topics are inseparable and should be addressed simultaneously.

Keywords: algebra; variable; K-12 education; theory of conceptual fields.

A construção do conceito de variável por estudantes da educação básica: contribuições Da Teoria dos Campos Conceituais

RESUMO

Contexto: Neste artigo apresentamos um recorte de uma pesquisa que visou a construção do conceito de variável por estudantes do 8º ano do Ensino Fundamental. **Objetivos**: O objetivo é analisar as estratégias e os argumentos utilizados por

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estudantes do 8º ano do Ensino Fundamental quando confrontados com o conceito de variável no estudo da ágebra. **Design**: Trata-se de uma pesquisa qualitativa em educação com características de um estudo de caso, em que realizamos uma intervenção de ensino que foi construída, vivenciada e analisada à luz da Teoria dos Campos Conceituais. **Ambiente e participantes**: Participaram da pesquisa sete estudantes da turma de 8º ano da escola pública de Duque de Caxias/RJ, onde a intervenção foi realizada. **Coleta e análise de dados**: Filmamos a intervenção e transcrevemos as falas dos participantes. A Teoria dos Campos Conceituais fundamentou a pesquisa oferecendo as noções de conceito, campo conceitual, esquema, representação, invariante operatório, álgebra e pensamento algébrico. **Resultados**: Os resultados mostram que os estudantes desenvolvem uma gama de estratégias e argumentos para lidar com a noção de variável. Além disso, apresentam dificuldades para lidar com expressões do tipo "mx + n". Os números inteiros e as operações com eles estão associados à construção do conceito de variável e a diversificação da letra que representa a variável pode ser um obstáculo quando não é bem compreendida pelos estudantes. **Conclusões**: A compreensão de expressões algébricas envolve a distinção entre os conceitos de incógnita e variável e, por isso, esses temas são indissociáveis, devendo ser abordados simultaneamente.

Palavras-chave: álgebra; variável; educação básica; teoria dos campos conceituais.

INTRODUCTION

In this text, our objective is to analyze the strategies and arguments used by 8th-grade students in elementary school when confronted with the concept of variable in the study of algebra. This is a segment of a broader research project that aimed to develop and analyze a teaching intervention that would contribute to the construction of the concept of variable by 8th-grade students from a public school located in a peripheral context. Some of the ideas presented here were discussed during the GT2 meetings at the IX Symposium on Research in Mathematics Education (Sipem), held in November 2024 in Natal, Rio Grande do Norte. GT2 is the working group of the Brazilian Society of Mathematics Education (SBEM) that gathers studies focused on mathematics in the final years of elementary school and in high school.

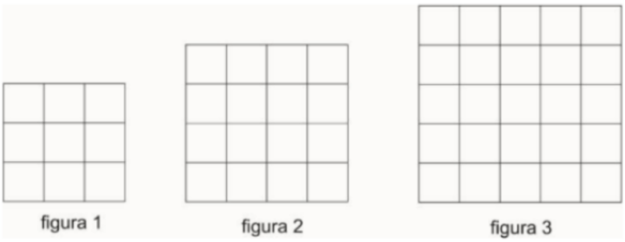
Many studies address algebra in the attempt to define it, and, in many cases—as in the experience reported here—students define it merely as the use of letters in mathematics. Borges (2018) also points out that many teachers present the idea of a variable as letters that represent numbers. However, Usiskin (1995, 1999) argues that algebra cannot be reduced solely to the use of letters. Based on this, the author presents four conceptions of algebra: algebra as generalized arithmetic; algebra as the study of procedures; algebra as the study of relations between quantities; and algebra as the study of structures (Usiskin, 1995, 1999).

In the conception of algebra as generalized arithmetic, algebra must arise from arithmetic situations as a movement of generalization. Students should interpret and generalize variables made explicit in situations, finding mathematical models. Figure 1 shows an example of this conception.

Figure 1

Generalization of sequences. (Guimarães,2013)

Observe as figuras a seguir, formada por linhas e colunas de quadradinhos:



a) Determine a quantidade de quadradinhos da figura 1 e da figura 2;
b) Qual a quantidade de quadradinhos da figura 3?
c) Quantos quadradinhos teria a figura 6?

In the conception of algebra as the study of procedures, the focus is on solving problems that involve simplifying and solving equations. Usiskin (1999) illustrates this conception with the problem: “When we add 3 to five times a number, we obtain 40 as a result. Find the number.”

The problem is translated into the following equation: $3 + 5x = 40$. According to the author, many students struggle in the transition from arithmetic to algebra, since the arithmetic solution would be to subtract 3 and divide by 5, while the algebraic form $5x + 3$ involves multiplication by 5 and the addition of 3. Therefore, to solve the equation, one must think in terms of inverse operations.

On the other hand, the conception of algebra as the study of relations between quantities differs from the previous one, since letters represent variables—that is, values that change. An example given by the author is the formula for the area of a rectangle, $A = B.H$, which does not require finding an unknown value but instead expresses a relationship between quantities.

In the fourth conception, algebra as the study of structures, there is no model to be generalized, no equation to be solved, and no relationship between variables. Usiskin uses as an example the factorization of $3x^2 + 4ax - 132a^2$, which results in $(3x + 22a)(x - 6a)$. In this case, the idea of variable does not coincide with any of the previous conceptions.

As a conclusion to his work, Usiskin (1999) asserts that the role of algebra goes beyond being merely an instrument for solving problems; it is fundamental for the characterization and understanding of mathematical structures. This fact, according to the author, justifies algebra as the main area of study in high school. The emphasis on algebra in high school, in turn, highlights the need for its introduction in the final years of elementary school, and the different ideas of variable present in the various conceptions of algebra led us to consider the concept of variable as one of the central ideas to be addressed at this stage. Therefore, in this study, our research question is: what are the strategies and arguments that 8th-grade students in elementary school use when confronted with the concept of variable in the study of algebra?

Throughout the text, the term "algebraic thinking" will be used more frequently than algebra. According to Fiorentini, Miorim, and Miguel (1993, p. 87), algebraic thinking is characterized by "the perception of regularities, perception of invariant aspects in contrast with those that vary, attempts to express or make explicit the structure of a problem situation, and the presence of the process of generalization."

Furthermore, according to Fiorentini, Miguel, and Miorim (1993), regarding the language through which algebraic thinking can be expressed, we can affirm that there is not only one form of language. It can be expressed through natural language, through arithmetic, geometry, or a specific language known as algebraic language, in a symbolic form. For this reason, these researchers argue that it is necessary to rethink the relationship between algebra education and algebraic thinking. It is necessary to go against the practice based solely on algorithms and create conditions for students to produce meanings for symbols and procedures, which should be investigated and justified. In this sense, the concept of variable plays a central role in the study of algebra, and algebra should be included from the early years through the generalization of arithmetic patterns, breaking the idea that algebra and arithmetic are separate parts of mathematics. Corroborating these statements, Bilhalva (2020) asserts that the distinction between algebra and arithmetic can lead to misconceptions for students:

> For example, when students encounter an expression such as x + 2, they tend to add the elements, combining them all as in arithmetic (obtaining 3x), as if it were possible to add a number with a literal part, since, for them, the equal sign implies a result (some even "eliminate" the symbol x, because it has no meaning for them) (Bilhalva, 2020, p. 24).

Thus, although in some circumstances algebra may be seen as the generalization of arithmetic, this is not the only way to approach algebraic content. It is necessary to take into account different aspects of these areas of mathematics, since arithmetic seeks to find concrete solutions, while algebra deals with generic situations. To develop algebraic thinking, students must be able to investigate regularities, systematize properties, solve and discuss algebraic problems, model situations, and determine patterns among different pieces of information.

It is important to mention that algebra is one of the five thematic units suggested by the Brazilian National Common Curricular Base (BNCC) for the teaching of mathematics at all stages of basic education. The thematic unit "Algebra" aims to develop algebraic thinking, which, according to the BNCC, "[…] is essential to use mathematical models in understanding, representing, and analyzing quantitative relationships of magnitudes, as well as mathematical situations and structures, making use of letters and other symbols" (Brazil, 2017, p. 270).

Defining equivalence, variation, interdependence, and proportionality as fundamental mathematical ideas, the document states that some dimensions of algebra must be addressed from the early years. For this stage, the proposal is to develop the ideas of generalization, regularity, and equality properties, but not the use of letters, which is introduced only in the later years. An example given is the use of activities involving equality, such as 2 + 3 = 4 + 1, so that students understand that the equal sign is not merely an indication of an operation to be performed (Brazil, 2017). In this way, the document reinforces the notion that starting the study of algebra in the early grades is essential to break the conception that algebra is merely generalized arithmetic.

As previously mentioned, the group of participants in this research consists of students in the final years of elementary school, specifically 8th graders. The BNCC presents some considerations for this stage of learning, stating that schools should provide students with teaching aimed at grasping the meaning of mathematical objects. The document emphasizes that "[…] at this stage, the importance of communication in mathematical language with the use of symbolic language, representation, and argumentation needs to be highlighted" (Brazil, 2017, p. 300).

It is also worth mentioning that the Municipal Department of Education (SME) of Duque de Caxias, the city where this research was conducted, organized, inspired by the BNCC, at the end of 2022, a curricular restructuring for Early Childhood Education, Elementary Education, and Youth and Adult Education (EJA). For all these educational levels, the document presents guidelines, updating the vision of education and its objectives (Duque de Caxias, 2022).

Specifically for the final years of elementary education, the document states that the teaching of mathematics should aim to provide students with training that fosters a positive view of the discipline, highlighting its importance for society alongside other sciences. Like the BNCC, the organization of mathematical content is divided into five thematic units: numbers, quantities and measurements, algebra, geometry, probability, and statistics (Duque de Caxias, 2022).

Regarding the thematic unit "Algebra," the curriculum framework presents algebraic thinking as its main objective, "with the goal of understanding and representing relationships of magnitudes, equivalences, variation, interdependence, and proportionality" (Duque de Caxias, 2022, p. 75). The integration of these contents should lead students to the perception of regularities, patterns in numerical and non-numerical sequences, the interpretation of graphical and symbolic representations, as well as the solving of equations and inequalities.

The curriculum framework was developed as a pathway for students to develop the skills and competencies contained in the BNCC. To this end, the document stresses the need for appropriate methodologies and qualified teachers. Among the specific objectives to be achieved in the later years of mathematics, one stands out: "[…] to develop algebraic thinking as the mathematical generalization of arithmetic and as an expansion of the possibilities of argumentation and problem solving" (Duque de Caxias, 2022, p. 79).

Table 1 shows part of the mathematics curriculum matrix for 7th and 8th grades. It focuses on algebra content. The matrix topics were taken from the BNCC, identified by alphanumeric codes, as the document indicates.

It is important to highlight that the matrix demonstrates the concern with developing algebraic thinking from the early years, since, as previously mentioned, it follows the BNCC guidelines. Thus, starting from the 1st year of the Literacy Cycle, there is the “Numbers and Algebra” strand, which includes knowledge such as “[...] identifying implicit rules and patterns in recursive or repetitive sequences, whether numerical, figurative, involving objects, sounds, etc.” (Duque de Caxias, 2022, p. 80).

Table 1

Algebra curriculum framework for 7th and 8th grade. (Duque de Caxias, 2022, p. 94; p. 103)

7th grade	8th grade
(EF07MA13) Develop an understanding of the concept of a variable, represented by a letter or symbol, as a way to express the relationship between two quantities, and distinguish it from the concept of an unknown.	(EF08MA06) Solve and create problems that involve calculating the numerical value of algebraic expressions, using the properties of operations.
(EF07MA14) Classify sequences as recursive or non-recursive, recognizing that the concept of recursion is present not only in mathematics but also in arts and literature.	(EF08MA07) Associate a first-degree linear equation with two unknowns to a straight line in the Cartesian plane.
(EF07MA15) Use algebraic symbols to express patterns found in numerical sequences.	(EF08MA08) Solve and create problems related to real-life contexts that can be represented by systems of first-degree equations with two unknowns, and interpret them, including through the use of the Cartesian plane.
(EF07MA16) Recognize whether two algebraic expressions describing the same sequence are equivalent or not.	(EF08MA09) Solve and create problems, with and without the use of technology, that can be represented by second-degree polynomial equations of the form $ax^2 = b$.
(EF07MA17) Solve and create problems involving direct and	(EF08MA11) Identify the pattern of a recursive numerical sequence and

inverse proportionality between two quantities, using algebraic statements to express the relationship between them.	construct an algorithm, through a flowchart, that allows determining the following numbers.
(EF07MA18) Solve and create problems that can be represented by first-degree polynomial equations, reducible to the form $ax + b = c$, applying the properties of equality.	(EF08MA12) Identify the type of variation between two quantities—direct, inverse, or non-proportional—express the relationship through an algebraic statement, and represent it in the Cartesian plane.

Analyzing the curriculum matrix shows that, in 7th grade, students are expected to recognize regularities and sequences, understand and distinguish between variables and unknowns, and grasp equality through first-degree equations. This focus demonstrates the municipality's commitment to developing students' algebraic thinking so they can deepen their understanding in the 8th grade.

As this study focuses on constructing the concept of variable, the next section reviews research on teaching algebra in the final years of elementary school. Subsequent sections cover the cognitive theory of learning that underpins our analysis, the research methodology, the results, and final considerations.

RESEARCH ON ALGEBRA IN THE FINAL YEARS OF ELEMENTARY EDUCATION

To ground our study, we conducted a review of Brazilian research produced between 2018 and 2023 that addresses the teaching of algebra in the final years of elementary education.

In the article Recurso lúdico para apoio ao aprendizado da álgebra de alunos do 7º ano do Ensino Fundamental, Serpa and Kinast (2021) aimed to analyze the effectiveness of a playful pedagogical activity on algebraic content with a 7th-grade class. As theoretical foundations, the authors drew on ideas related to the teaching of mathematics, the teaching of algebra, and the use of playful pedagogical resources. They also discussed aspects of Ethnomathematics, which emphasizes mathematics education grounded in students' social realities.

The study, qualitative and exploratory in nature, was conducted in three stages. In the first, students were asked to construct mathematical sequences using challenge cards; in the second, they received new cards with simple equations; and in the third, they solved first-degree equations. In all stages, the cards incorporated playful elements such as drawings of fruits and objects. Serpa and Kinast (2021) concluded that playful resources oriented toward algebraic thinking—such as puzzles and riddles—contributed to meaningful learning. They also emphasized the importance of contextualizing algebraic content, since students must learn beyond formulas and procedures.

Souza's (2021) dissertation, O estudo de álgebra no ensino fundamental II: Uma proposta com materiais manipuláveis, sought to explore the contributions of using manipulatives in teaching algebra to 8th-grade students. In her literature review, the author highlighted research that shows the potential of manipulatives as tools for visualizing abstract concepts. The concepts addressed during the intervention included algebraic language, algebraic expressions, the numerical value of expressions, and first-degree equations. The teaching materials developed were the "expression board" and "algebra tiles." Their use led to improvement in students' understanding of algebraic concepts, as reflected in better performance on subsequent assessments.

Righi, Dalla Porta, and Scremin (2021), in their article Pensamento algébrico: Uma análise de livros didáticos dos anos finais do Ensino Fundamental, published in the Revista Eletrônica de Educação Matemática, examined whether recursive sequences—part of the "Algebra" unit in the BNCC—are addressed in textbooks. They analyzed 8th-grade textbooks from the collections of Edwaldo Bianchini (Matemática – Bianchini: Manual do Professor) and Gelson Iezzi (Matemática e Realidade), both published in 2018 and approved in the PNLD/2020. Their findings indicated that sequence-related content was included, though not always in specific chapters, but rather distributed across algebraic topics. Overall, they concluded that the textbooks align with official guidelines and support the development of algebraic thinking.

Reis, Silva, and Santos (2021), in the Brazilian Electronic Journal of Mathematics, published Educação algébrica: O uso de padrões figurativo-numéricos como recurso didático-pedagógico para os anos finais do Ensino Fundamental, with the aim of identifying the contributions of figurative-numeric patterns as a teaching resource. The authors highlighted that the BNCC (Brazil, 2017) values the development of algebraic thinking as fundamental to problem solving. They advocated for the inclusion of pattern-based activities, as these communicate regularities that can first be observed and later generalized. Their qualitative, bibliographic study concluded that figurative patterns enhance the development of algebraic thinking.

Anjos's (2021) dissertation, Equações do 1º grau: Significando a aprendizagem por intermédio da história da matemática, proposed a teaching approach for a 7th-grade class grounded in the history of mathematics and David Ausubel's theory of meaningful learning. The teaching sequence was structured around the three historical stages of algebra: rhetorical, syncopated, and symbolic. According to the author, this approach allows students to "find in the oral and written language of rhetorical algebra the explanations for the algebraic symbols established throughout history" (p. xx). The study also discussed the importance of teacher education, suggesting that teachers themselves can benefit from historical perspectives, both in classroom practice and in deepening their own understanding of mathematics.

Silva's (2023) dissertation, Sequência didática como estratégia de ensino e aprendizagem para o desenvolvimento do pensamento algébrico nos anos finais do ensino fundamental, examined the impact of a teaching sequence addressing polynomials and their operations in a 9th-grade class, framed by the BNCC. The study drew on theoretical perspectives concerning the use of teaching resources in the final years of schooling, the challenges of using concrete materials, and conceptions of algebraic thinking. Using a qualitative case study methodology, Silva found that structured teaching sequences, combined with problem-solving strategies and instructional materials, increased students' engagement and supported the development of algebraic skills.

Taken together, these studies provided valuable insights for the present research. Discussions of official curricular documents reinforced our decision to work with 8th-grade students. While algebra—and specifically the concept of variable—does not begin exclusively at this stage, it is here that opportunities arise for a broader approach that incorporates Usiskin's (1999) conceptions of variable. Moreover, research highlighting the role of teaching resources guided our choice of materials for classroom activities. In line with these findings, we placed emphasis on manipulatives and activities that foster collective or small-group reflection, making students' strategies and reasoning visible as they engage with the concept of variable.

Finally, Scremin and Rigui's (2020) article "Ensino de álgebra no ensino fundamental: Uma revisão histórica dos PCN à BNCC" provided a historical review of curricular guidelines for algebra instruction. Through a documentary analysis of the Parâmetros Curriculares Nacionais (PCNs) and the

BNCC, they concluded that, despite being formulated over twenty years ago, the PCNs remain relevant. The PCNs recommend introducing algebra content from 7th grade onward, based on students' readiness for logical connections and abstraction, whereas the BNCC proposes algebra instruction from the early grades, emphasizing the observation of regularities, generalization of patterns, and properties of equality.

THEORY OF CONCEPTUAL FIELDS

The Theory of Conceptual Fields was proposed by Gérard Vergnaud (1933–2021), a French psychologist. He described it as "[…] a cognitivist theory that aims to provide a coherent framework and some basic principles for the study of the development and learning of complex skills, particularly those relevant to the sciences and techniques." (Vergnaud, 1990, p. 135).

Vergnaud states that the TCF is not limited to mathematics. It can be applied to any discipline where the objective is learning. Moreover, he acknowledges that it is not a simple theory, as he explains:

> [...] it involves the complexity that comes from trying to encompass, within a single theoretical perspective, the entire development of situations that are progressively mastered. It also includes the concepts and theorems necessary to operate in these situations and the words and symbols that represent these concepts and operations for students, according to their cognitive levels. (Vergnaud, 1994, p. 43).

According to the definitions, a conceptual field can be understood as a set of situations. For example, the conceptual field of additive structures encompasses situations that require addition, subtraction, or a combination of both operations. The conceptual field of multiplicative structures involves situations that require multiplication, division, or a combination of the two. Working with the notion of situation allows for the construction of a classification based on the analysis of cognitive tasks and the procedures required for each (Vergnaud, 1993).

Other key concepts in the theory include situations, schemas, implicit operative invariants (theorems-in-action and concepts-in-action), and explicit ones. Vergnaud argues that mastering the knowledge of a conceptual field requires time, experience, maturity, and learning. Therefore, overcoming a conceptual difficulty does not happen overnight. Vergnaud (1993) maintains that a concept cannot be reduced to its definition and must instead be represented by a triad (S, I, R):

- Situations (S): The set of situations that give meaning to concepts (a combination of tasks);
- Invariants (I): The set of invariants that form the properties of the subjects (the meaning);
- Representations (R): The set of symbolic representations used to express situations and procedures (the signifier).

From this triad, it is possible to understand aspects of the learning process, as one must consider that a concept is not formed in a single situation and that a situation cannot be analyzed through only one concept (Vergnaud, 2009).

Regarding situations, Vergnaud (1993, p. 1) states that it is "through situations and problems to be solved that a concept acquires meaning for the child." These can be divided into two classes:

a) classes of situations in which students already possess the necessary skills to solve them immediately; and

b) classes of situations in which students do not yet possess all the required skills, which demand learning time and may involve both successes and failures along the way.

It is essential to note that, according to this theory, the concept of situation does not refer to a didactic situation but rather to a task, as Vergnaud (1993) explains. The author points out that the difficulty of a task "is neither the sum nor the product of the difficulty of its subtasks. It is clear, however, that failure in a subtask causes overall failure." (Vergnaud, 1993, p. 9). Every complex situation is understood as a combination of certain tasks, each with its own nature and level of difficulty that must be well understood.

For both classes of situations mentioned above, the use of schemas is necessary, although their functioning differs in each case. Schemas were first introduced by Piaget as a way to understand how sensory-motor and intellectual skills are organized. From that perspective, the focus was on the epistemic subject, that is, on the investigation of the major categories of thought: space, time, causality, etc. (Silva & Frezza, 2011). Vergnaud (1993) adopts the concept of schema but believes the focus should be on the subject in action. A schema should be composed of rules and can be effective across multiple situations, generating different actions (Barbosa, 2008).

> [...] [Vergnaud] borrowed from Piaget important aspects of his work: first, the concept of schema, which has a broad

> interpretation, the idea that knowledge is adaptive (accommodation and assimilation), as well as Piaget's global conception that action and representation are part of development. (Vergnaud, 2009, p. 84).

Regarding the use of schemas in his theory, Vergnaud (1993, p. 2) defines them as "*the invariant organization of behavior for a given class of situations.*" They are composed of the subject's knowledge-in-action, that is, the cognitive elements that make an action operational. A classic example is the schema for solving equations of the form $ax + b = c$. For this type of equation, when the values of a, b and c are positive and $b < c$, the schema quickly reaches a high degree of availability and reliability among beginning algebra students. The solutions presented by students reveal an invariant organization of what they have learned from theorems, such as subtracting "b" from both sides to maintain equality or dividing both sides by "a" to preserve it.

Vergnaud (1990) states that schemas are logical devices of the same type as algorithms, which may or may not be sufficient for a given situation. Schemes are often effective, but not always successful. When a schema becomes ineffective, experience may lead the student to seek a new schema to achieve their goal.

The algebraic conceptual field can be defined as the set of situations, representations, and invariants necessary for the construction of algebraic concepts (Klopsch, 2010). Recognizing the schemas required for this field is fundamental to analyzing the difficulties students face in algebra. Vergnaud (2019) asserts that, based on arithmetic knowledge, algebra represents a major formal shift and exhibits characteristics that distinguish it from arithmetic. Thus, Table 2 presents these differences.

Table 2

Differences between arithmetic and algebra. (Vergnaud, 2019)

Arithmetic	Algebra
intermediate unknowns	extraction of relevant relationships
intuitive selection of data	formal expressions of statements and operations
operations in the correct order	algorithm
guided by meaning	control: rules and appropriate model

According to the author, to operate in algebra, it is necessary to have a "script-algorithm," such as the process of solving an equation. When solving an equation, although simple arithmetic operations are involved, students often face many difficulties because they still need to develop new competencies. These competencies represent a rupture with arithmetic. Vergnaud (2019) presents them as follows:

> 1- Knowing what to do when faced with a given equation and achieving a certain goal while respecting the rules. 2- Knowing how to translate a problem into an equation, extract the relevant relationships, and control their independence. 3- Identifying new mathematical objects—equation and unknown, function and variable. 4- Recognizing the role of algebra—to solve complex problems and to prove relationships. (Vergnaud, 2019, p. 17)

These competencies encompass different levels of conceptualization. The first two are based on Piaget's schemas, the third is grounded in explicit conceptualizations, and the fourth is metacognitive (Vergnaud, 2019).

To illustrate the use of schemas, Kikuchi (2019) presents a table with the schemas that must be mobilized for the main distributive property content within the conceptual field of algebraic structures. The author states that "schemas generate a class of behaviors associated with a specific situation, functioning as an organizer of thought" (Kikuchi, 2019, p. 68), and for each schema, it is possible to identify the doubts expressed by students. Thus, Table 3 presents a summary connecting the content, the conceptual field, and the corresponding schemas.

Table 3

Summary table linking content, conceptual field, and schemas. (Kikuchi, 2019, p. 130 – Modified)

Multiplication of two identical algebraic terms	**Confuse $a·a = a^2$ and representing it *as* $a·a = 2a$**
Addition of different algebraic terms	Add $a + b$ and obtaining ab as a result
Addition of identical algebraic terms	Confuse the addition of identical terms such as $ab + ab$ and multiplying them, resulting in a^2b^2

Multiplication involving the sum of two algebraic terms	Apply only the exponents to the terms in parentheses, e.g., $(a + b)^2 = a^2 + b^2$
Commutativity in the multiplication of two algebraic terms	Not recognize that $a \cdot b = ab$ and $b \cdot a = ba$
Commutativity in the multiplication of two algebraic terms	Understand that the content inside the parentheses should be treated as a single term
Commutativity in the multiplication of two algebraic terms	Believe that the coefficient of a variable x is always 1

The Theory of Conceptual Fields seeks to work with situations in which concepts begin to make sense to students. Vergnaud (2019) points out the difficulty students face when dealing with integers, as they often believe they have made a mistake when they obtain a negative result after solving an equation. One alternative is to use everyday situations in which negative numbers appear, such as temperature, scores, and debts, among others.

METHODOLOGY

Regarding the type of research, the study presented here can be classified as qualitative research, which, according to Minayo (1994),

> […] works with a universe of meanings, motives, aspirations, beliefs, values, and attitudes, corresponding to a deeper level of relationships, processes, and phenomena that cannot be reduced to the operationalization of variables. (Minayo, 1994, p. 21).

The choice of this research approach is justified by the objective of the investigation: to analyze the strategies and arguments used by 8th-grade students in an Elementary School when confronted with the concept of variable in the study of algebra. We conducted a teaching intervention, and regarding this type of approach, Gil (2002) states that

> Qualitative analysis depends on many factors, such as the nature of the data collected, the size of the sample, the research instruments, and the theoretical assumptions guiding the investigation. However, this process can be defined as a sequence of activities involving data reduction, data categorization, interpretation, and report writing. (Gil, 2002, p. 133).

The research was carried out with 7 out of 42 students from an 8th-grade class at a state public school located in the Municipality of Duque de Caxias/RJ, during the second semester of 2023. Among other reasons mentioned earlier, this grade level was chosen because students begin to engage with algebra starting in the 7th grade. Therefore, we expected them to be able to discuss this content meaningfully. The selection of the seven participating students took into account their attendance, punctuality, and interest. All of them were regular, participative, and highly engaged students.

Since this study was conducted within a specific context — a group of students from a particular 8th-grade class — the research can also be characterized as a case study (Lüdke & André, 1986). This methodological approach considers the complexity involved in the research context, as each subject is unique. According to the authors, the case study directly involves the researcher in the situation in order to obtain information, emphasizing the process rather than the product, and taking into account the participants' perspectives.

Information was gathered through a teaching intervention lasting 100 minutes, which consisted of playing the Algebraic Race game and engaging in a collective reflection about the game. It is important to note that the collective reflection, which took place after the students had played several rounds, was also part of the intervention.

In addition to using dice and tokens for each team's progress on the board, the resources used included the game board and cards, shown respectively in Figures 2 and 3 below.

Figure 2

Game board of the Algebraic Race. (The authors, 2023)

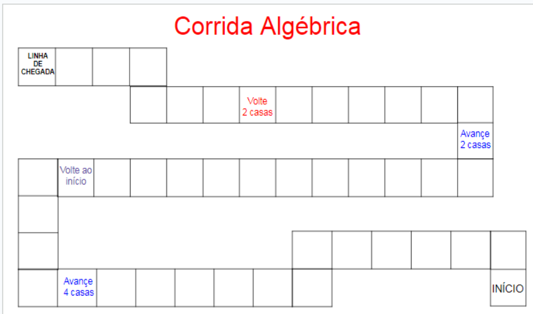


Figure 3

Cards from the algebra race game. (The authors, 2023)

$x + 2$	$x - 4$
$2x + 1$	$5 - x$
$3 + x$	$2x - 3$
$2 - x$	$2a$
$1 + 2a$	$6 - a$

The class was divided into pairs or trios. Each received a game board, ten cards, and a token to identify their team. Each card had a different algebraic expression (e.g., $x + 1$, $2x - 4$, $4 - x$, etc.). The cards were stacked face down. Students drew one card at a time. The lead teacher and the graduate student, who conducted the intervention together, rolled a die to determine the value to apply to the algebraic expression. This value showed the number of spaces the group could move forward or backward. For example, if the number rolled was 4, Group 1 with $(x + 1)$ would advance 5 spaces. Group 2, holding $(2 - x)$, would move back 2 spaces. The group that reached the end of the board path first won the game.

After playing the game, the class participated in a collective reflection guided by the following questions:

- Did you have any doubts during the game?

- Were these doubts resolved? How?

- Should we create new rules for the game?

- Imagine that the number on the die represents the value that the expression should be. The value of the variable that makes the expression equal to the die's number shows how many spaces your token moves. How would you calculate the steps to move your token?

To play, the seven participants in this study formed one trio and two pairs. The entire teaching intervention was recorded, and the participants' statements were transcribed. The names mentioned in the following section are fictitious to preserve the identity of all involved. Considering that the proposed activities were part of the regular classroom routine, approval from ethics committees was not required. However, all participants and their guardians, as well as the class teacher and school administrators, were previously informed about all stages of the research. The Informed Consent Form (ICF) signed by them is included as supplementary documentation to this article.

RESULTS AND ANALYSIS

The objective of the game used in the teaching intervention was to create conditions for students to understand the concept of a variable in an algebraic expression. This variable could assume different values, determined by the face of the die showing upward after a roll. The challenge was selected based on students' responses in the diagnostic test, in which a considerable portion of the class reported enjoying games and stated that they usually play in their free time.

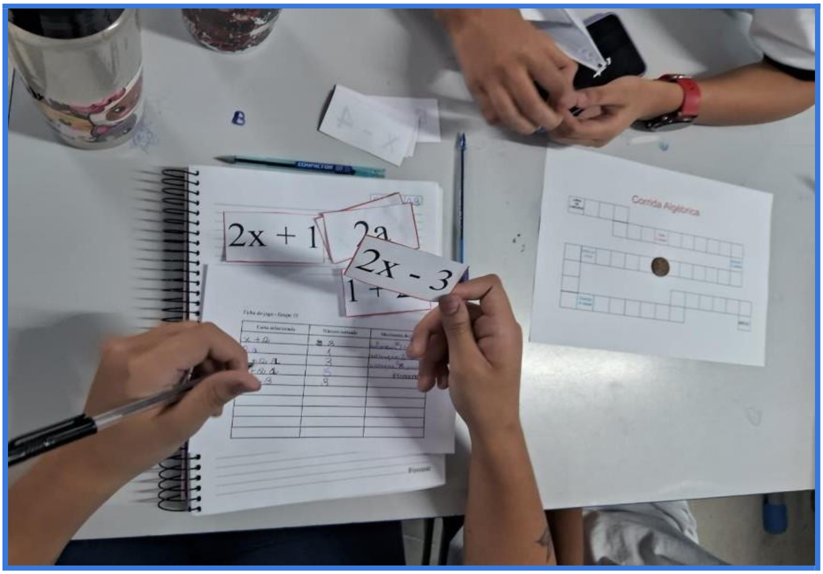
The intervention began with dividing the class into trios or pairs. It was observed that the students had difficulty interacting, as many classmates did not communicate with each other. As a result, 10 trios and 3 pairs were formed. During this stage, despite some restlessness, several students showed curiosity about what would take place. After the division, the materials were distributed, which generated even more curiosity, since, although all students were familiar with board games, they did not know how the cards with algebraic expressions would be used. Despite being instructed to keep the cards stacked and face down, some groups, driven by curiosity, looked at the content in advance.

In general, students demonstrated interest in the activity from the start, and as soon as they received the materials, they began organizing themselves to divide roles, such as who would manage the board, who would fill in the table, and who would handle the cards.

After the class was organized and the materials distributed, the graduate student began explaining how the game worked. However, after the first explanation, most students reported not understanding the proposal, making a second explanation necessary. During the first attempt, the restlessness in the classroom made it difficult to hear the instructions. At this point, intervention from the lead teacher was required to calm the students and get their attention. After the second explanation, the class generally understood the activity. Nevertheless, most students still had difficulties performing the calculations involving the algebraic expressions, which required the graduate student's intervention at various moments. Figure 4 illustrates a pair during the development of the game.

Figure 4

Playing the Algebra Race Game. (The authors, 2023)



After the explanation and once most of the class had focused, the graduate student positioned herself in front of the teacher's desk and asked the trios/pairs to draw their first card. After doing so, she rolled the first die and obtained the number 3. Since the cards were shuffled, each trio/pair drew a different card from the 10 available. As there were 13 groups, some drew duplicate cards. Many students did not know what to do with the number 3 rolled, so the graduate student explained that they should substitute the number in place of the variable and perform the operation.

The trios/pairs that initially drew the cards "x + 2" and "x + 3" asked the graduate student for help, as shown in the transcript below:

B.: Teacher, so we should put 3 in place of "x"?

M.: Exactly. What is the result of this calculation?

(Students think for a moment.)

L.G.: Should we add 3 and 2?

M.: That's right. This will be the number of spaces you should move forward.

B.: So, we move 5 spaces, then, guys.

M.: Yes, but you also need to record it on the table.

B.: What do we write for the game movement?

M.: You reached the number 5, and since it is positive, you can write "I've advanced 5 spaces."

The students moved forward five spaces and showed enthusiasm for the next turn. According to Vergnaud (1993), the reliability in mobilizing a schema is based on the knowledge they possess, whether implicit or explicit, regarding the relationships between the algorithm and the attributes of the problem to be solved. In general, students who drew cards with expressions of the type "x + b" did not show much difficulty performing the calculations. Once their doubts were resolved, they were able to continue the game with more confidence, as seen with the students who drew the cards "x + 2" and "x + 3."

Difficulties arose more frequently when the drawn card was of the type "*mx* + *n*." Most of the class could not continue the game when they drew this type of card. It was observed that students did not understand that structures like "*2x*" and "*2a*" represent multiplication between the constant number and the value assigned to the variable. Therefore, the graduate student's intervention was necessary to assist them. The following transcript, involving the pair J and A, who drew "*2x + 3*" as their first card, illustrates this point:

J.: Teacher, what do we do with this “2” in front of the x?

M.: When we have a number “stuck” to a letter, there’s an operation between them.

J.: So it becomes 23?

M.: No, when there’s a number in front of a letter, it means there’s a multiplication between them.

The students made gestures and facial expressions showing confusion. The graduate student then moved to the front of the class and asked everyone to pay attention. At that moment, a brief explanation on the board about the structure of an algebraic expression was necessary. She explained to the class that there is a multiplication operation between the number and the letter. An example was given using the expression “$2x + 1$” with the value $x = 1$.

After that, the class continued with the game, and the graduate student returned to the pair J and A:

J.: Teacher, so we multiply 2 by 3 and then add 1?

M.: Exactly. And what’s the final result?

(The students think together, vocalizing the calculations they are doing.)

A.: 7?

M.: That’s right.

Similarly, other groups asked for help to calculate expressions of the form $mx + n$, but it was observed that the groups playing later rounds and drawing cards of this type no longer had as much difficulty.

Based on the two classes of situations proposed by the Conceptual Fields Theory (CFT), we understand that the use of schemes is not limited only to the first class. Vergnaud (1993) states that when students are unable to solve a problem or make mistakes, they are in fact also mobilizing schemes, which must be observed by the teacher. The pair J and A mobilized the following schema: “When a number is next to a letter and the letter takes on a value, substitute the letter for the value and place it next to the number.”

From that point on, the game progressed more smoothly, and only occasional doubts arose. One major difficulty presented by some students was dealing with negative values. For example, when the number 3 was rolled, a pair who had the card ($x – 4$) could not correctly calculate the result, prompting a discussion on the topic. Seeing that other students had the same doubt, the graduate student again stood at the front of the class and asked everyone to pay attention. The following is a transcript of the conversation that took place:

M.: Class, the number 3 was rolled, and some of your classmates have the card "*x - 4*." Does anyone know the result?

N.: You can't calculate that. You can't take 4 away from 3.

B.: It's -1.

C.: It's 1.

M.: Let's think about it for a bit. I think you've already studied integers at some point, but it really takes time to understand them well.

At that moment, the main teacher said that integers had been covered the previous year and reminded the class about negative numbers. After her comment, many students began recalling what they had learned and started answering "-1."

M.: That's right, everyone, three minus four equals -1. But going back to the game, what will happen to the group that got this result?

Class: They'll have to move back one space.

One of the prerequisites for the Algebra Race Game is the concept of integers, which the students are still developing. Vergnaud (1993) emphasizes in the Theory of Conceptual Fields (TCF) that it is not possible to build concepts in isolation or linearly. Therefore, the use of fundamental arithmetic concepts is necessary to develop competence in the algebraic conceptual field.

Continuing the game, the second number rolled was 1. Up to that point, no group had drawn the cards "2a," "6 - a," or "1 + 2a," but on the second roll, one pair drew the card "2a." The graduate student observed how the students would react, as they didn't know what to do when they saw a variable different from x. She approached them and began the following conversation:

R.: Miss, there's no "x" on this card, and we don't know what to do now.

M.: The "a" plays the same role as the "x."

R.: So we put the number 1 in place of "a"?

At that moment, the other student in the pair took the paper and wrote the expression, substituting the number 1 for “a.” After performing the multiplication, they got the result 2.

The two students looked at each other and said together:

R. and F.: We’re moving forward two spaces!

Similarly, other groups showed the same difficulty with the cards “*6 - a*” and “*1 + 2a*,” probably due to the shifted terms and the change of variable. For the card “*6 - a*,” however, there was no difficulty applying the rolled values, since they ranged from 1 to 6 and did not result in negative numbers.

We based our analysis on the triplet (S, I, R), which, according to Vergnaud (1993), constitutes a concept, and we sought to identify in the students’ schemes the manifestation of operational invariants resulting from the change of variable from x to a. In addition, we aimed to observe the importance of negotiating the meaning of symbols. After all, introducing a new letter without prior discussion caused estrangement among the students.

Despite the initial difficulties, our dialogical approach—alternating between algebraic symbols, natural language, and even gestures—helped guide the game and support student learning.

Moving on to the reflection phase of the game, when we asked the students about the doubts they encountered during play, we found that they were aware of the entire process we just described and could identify the difficulties they faced when dealing with negative numbers and when calculating expressions of the type mx + n. However, the understanding that they should proceed in the same way when the variable was represented by the letter a was not mentioned as a difficulty but rather as something that needed clarification—because, as student M said, it was something “*they couldn’t have guessed*.”

This statement, which was quite common among the students, suggested to us that they were indicating—albeit without a specific vocabulary—the need to negotiate the meanings of symbols, an important stage in the process of constructing algebraic thinking, as stated by Vergnaud (2019). Failing to understand this meaning is not equivalent to having learning difficulties. The negotiation of meanings is, in fact, an essential step in the learning process and may prevent future difficulties.

According to the students, overcoming their difficulties occurred through attentive listening to the teacher’s and researcher’s explanations and through the repetition of procedures in each round of the game. Regarding the new rules, we were struck by the students' desire to eliminate the possibility of moving backward on the game board. They recognized that moving backward occurred when the result of their calculation was a negative number, and at that moment, it was possible to reflect on the concept of the absolute value of an integer:

M: Is there a way to prevent a negative result?

N: No, because you do the calculation and get whatever result it gives.

M: But then, how can we avoid moving backward?

N: We can take the result without the sign, always making it positive, and move only forward.

It is worth noting here that, as predicted by Vergnaud (2019), algebraic thinking is linked to integers, and in this case, a game designed to build the concept of a variable, in addition to mobilizing operations with integers, also led to another important concept in the field of numbers—the concept of absolute value. Thus, we have yet another example that a concept is not constructed in isolation.

Finally, in the last guiding question, the students had the opportunity to compare the concept of a variable with that of an unknown. We returned to the game cards and asked:

M: If you roll a 6 on the die and get the card "2a + 1," how many spaces will you move?

All: 13.

M: With this card, will you always move 13 spaces?

R: No, it depends on the number that comes up on the die.

M: Now, with this new rule, if you roll a 5, how many spaces will you move?

R: Two.

M: How did you figure that out?

R: Since everything was the opposite of what we were doing before, I did the calculations the other way around, too.

We repeated these questions, assuming different numbers on the die, and the students were encouraged to find the single value that the unknown could take under this new rule. We took this opportunity to discuss the equality sign "=" and the importance of understanding it. The reversibility of operations, which underlies R's procedure, was also mentioned. We inferred that this was implicit knowledge in his action—theorems-in-action, as defined by Vergnaud (1990). Thus, within the context of the game, the students began the process of distinguishing between variables and unknowns. The procedures for solving equations were the topic of another intervention; however, according to Vergnaud (2019), distinguishing what is and what is not a variable is fundamental to constructing this concept.

CONCLUSIONS

In this study, our objective was to identify how 8th-grade students construct the concept of a variable. To this end, we conducted a game in which they were required to deal with the substitution of variables by integers in algebraic expressions.

Analyzed in light of the Theory of Conceptual Fields, the results show that students had greater difficulty working with expressions of the form mx + n than with those of the form $x + b$. This occurs because the schema used to understand the first type is more complex than the one employed for the second.

In both cases, it was possible to identify the mobilization of the concepts of integers and their operations, which constitute the students' knowledge-in-action. Furthermore, we observed that varying the letter used to represent the variable can become an obstacle when not properly discussed with the students—an aspect that, within the Theory of Conceptual Fields, corresponds to the negotiation of the meaning of representations.

In the final moments of the intervention, as we collectively reflected on the game, we concluded that understanding the concept of a variable necessarily involves distinguishing between the concepts of "unknown" and "variable." Therefore, we consider these topics to be inseparable and should be addressed simultaneously.

It is evident that, as this is a case study, the results cannot be generalized. Nevertheless, we hope they contribute to reflections on the construction of algebraic thinking.

AUTHORS' CONTRIBUTIONS STATEMENTS

GSB and DRSJ conceived the presented idea. GSB developed the theoretical framework. DRSJ adapted the methodology to this context, created the models, carried out the activities, and collected the data. GSB and DRSJ analyzed the data. All authors actively participated in the discussion of the results, reviewed, and approved the final version of the manuscript.

DATA AVAILABILITY STATEMENT

The data supporting the results of this study will be made available by DRSJ upon reasonable request.

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