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From fearing errors and misconceptions to discussing them: mathematics teachers' practice in a lesson study

Filipa Faria[a]
João Pedro da Ponte[a]
Margarida Rodrigues[b]
Marisa Quaresma[a]

[a] UIDEF, Instituto de Educação da Universidade de Lisboa, Portugal
[b] Ci&DEI, Escola Superior de Educação, Instituto Politécnico de Lisboa,

ABSTRACT

Background: Leading mathematical whole-class discussions remains a challenge for in service teacher practice. Lesson study is a collaborative professional development process based on real and current contexts of practice, which may contribute to face this challenge. **Objectives**: This study aims understand how lesson study may contribute to teachers' practice regarding whole-class discussion by identifying what productive networks link teachers' planning to lead and reflect on whole-class discussion. **Design**: This study follows an interpretative and qualitative approach. **Setting and Participants**: It was carried out in Portugal with three in service teachers, who planned and led a lesson for grade 5. **Data collection and analysis**: Data collection included audio and video recordings of the planning sessions, research lessons and post-lesson discussion and reflection, which we analysed by discourse analysis. **Results**: These discursive marks are associated with these productive networks: sharing challenges, facing challenges, sharing knowledge, accessing students' mathematical activity, actively listening, working with tasks and, finally, planning the lesson. The third mark, lesson study's catalysts, is associated with facilitator attitude, group collaboration and the observation/reflection of the research lesson. **Conclusions**: Sharing challenges, facing challenges, and accessing students' mathematical activity through a diagnostic task were especially relevant. Overall, productive networks did not operate in isolation; rather, they were interrelated and mutually reinforcing.

Keywords: mathematics education; lesson study; teacher practice; whole-class discussion; productive networks

Corresponding author: Filipa Faria. Email: filipa.faria@edu.ulisboa.pt

Do medo dos erros e conceções alternativas à sua discussão coletiva: a prática de professores de matemática num estudo de aula

RESUMO

Contexto: A condução de discussões coletivas em matemática continua a ser um desafio para a prática de professores em serviço. O estudo de aula é um processo colaborativo de desenvolvimento profissional, baseado em contextos reais e atuais da prática dos professores, que pode contribuir para enfrentar este desafio. **Objetivos** Este estudo tem como compreender de que forma o estudo de aula contribui para a prática dos professores no que diz respeito à discussão coletiva, identificando que redes produtivas ligam o planeamento dos professores à posterior condução e reflexão da discussão. **Design**: Seguiu-se uma abordagem qualitativa e interpretativa. **Ambiente e participantes**: O estudo foi realizado em Portugal com três professoras em serviço, que planearam e conduziram uma aula para o 5.º ano de escolaridade. **Coleta e análise de dados**: A recolha de dados incluiu gravações áudio e vídeo das sessões de planeamento, das aulas de investigação e das discussões e reflexões pós-aula, analisadas por meio da análise de discurso. **Resultados**: Foram reconhecidas duas marcas discursivas durante o planeamento e a condução das discussões: (i) a atividade matemática dos alunos, especialmente os erros e conceções alternativas; e (ii) a estrutura da aula e as intervenções. Estas marcas discursivas estão associadas às seguintes redes produtivas: partilhar de desafios, enfrentar de desafios, partilhar de conhecimento, aceder à atividade matemática dos alunos, escutar ativamente, trabalhar com tarefas e, por fim, planear a aula. A terceira marca, (iii) catalisadores do estudo de aula, está associada à atitude da facilitadora, à colaboração do grupo e à observação/reflexão da aula de investigação. **Conclusões**: Partilhar desafios, o enfrentar desafios e o aceder à atividade matemática dos alunos através de uma tarefa de diagnóstico evidenciaram-se especialmente relevantes. De forma geral, as redes produtivas não operaram de forma isolada, pelo contrário, mostraram-se relacionadas e mutuamente reforçadas.

Palavras-chave: educação matemática; estudo de aula; prática letiva; discussões coletivas; redes produtivas

INTRODUCTION

Research about whole-class discussions in mathematics identifies practices and actions for preparing and leading it as well as challenges associated with it (Boerst et al., 2011; Faria et al., 2024; Stein et al., 2008). Different aims of the lesson (Bussi, 1991; Henning et al., 2012; Takahashi, 2021) and different types of communication (Brendefur & Frykholm, 2000) and question posing (Herbel-Eisenmann & Breyfogle, 2005) lead to different whole-class discussions. In this study, we view a whole-class discussion as the phase of the lesson where the work towards a mathematical aim is object of co-construction by teacher and students, with negotiation of mathematical meanings. Therefore, the lesson aim must be clearly defined, and the

communication must be fuelled by the interactions between students, promoted and guided by the teacher.

In mathematics lesson study, a professional development process focused on student learning, whole-class discussions are the heart of the lesson (Takahashi, 2021). In this process, a group of teachers work collaboratively to plan a research lesson in detail, which is taught and discussed (Fujii, 2018; Stigler & Hiebert, 2019; Takahashi, 2021). Stigler and Hiebert (2019) consider that the research lesson is the smallest unit of teaching that preserves all the critical ingredients of the teaching-learning system. We add that the whole-class discussion is the critical phase of the lesson. Takahashi (2021) considers that "to engage the students in discussion, the teacher should be like the conductor of an orchestra, leading the students to help each other" (p. 27), stating that teachers need support to deal with challenges and try different approaches. In lesson study, this support arises from the participants' collaboration and reflection.

Based on Foucault's (1980) lens of power, we consider that participants' interventions in a lesson study produces knowledge, since power "needs to be considered as a productive network which runs through the whole social body" (p. 119). Power is not negative, but instead as a productive network able to produce knowledge from individuals' relationships, therefore, power may be an expression of the practice of a group. Power is taken as productive networks of interventions and activities that produce knowledge, and it can be visible by tracking changes in discourse. To Foucault, power and knowledge coexist. Accordingly, Jorgensen and Phillips (2002) admit that power does not belong to individuals, but it spread in different social practices, including teaching and learning. In this sense, a whole-class discussion may be analysed as a dimension of the teaching-learning process, with the teacher as an agent in three phases: the planning of the discussion, its leading and reflection.

In this study, we aim to understand how lesson study may contribute to teachers' practice regarding whole-class discussion. We pose the following question: what productive networks link teachers' planning to lead and reflect about a whole-class discussion in a lesson study?

TEACHERS' PRACTICE AND LESSON STUDY

Teachers' practice concerns the activities that they carry out frequently, in socially organized contexts (Ponte et al., 2012). These practices are co-constructed with other actors, such as students, colleagues, school principals, facilitators and parents. Teachers' practice is one of the factors that most

influence the quality of mathematics teaching and learning. This practice may be organized in three main interconnected fields: teaching practice, professional practice at the institution, and teacher education practice (Ponte & Serrazina, 2004). Regarding teaching practice, Ponte and Serrazina (2004) highlight the tasks proposed, materials used and leading classroom communication. They consider that communication in the classroom is much more than clearly explaining the contents, stressing that "another approach is needed that emphasizes not the quality of the teacher's discourse, but the quality of the shared discourse of teachers and students and how mathematical meanings are interactively constructed in the classroom" (p. 11).

Therefore, teaching practice can be organized in two dimensions: i) planned practice, considering the interactions with their colleagues, school leaders, reading of official documents and formulating expectations for students' mathematical activity; and ii) enacted practice, leading a lesson, considering the interactions that emerge from their students' mathematical activity and classroom communication. Teachers' professional development processes must consider a practice-based approach to better support teachers to articulate their planned and enacted practice, narrowing the gap between how teachers intend to work with their students and how they really do it.

Lesson study is a process that can bring classroom practice to teacher professional development, supported by collaboration and reflection (Richit & Ponte, 2017). This professional development process, formed by a group of teachers and usually guided by an external facilitator (Lewis et al., 2016), is "practice-based, research-oriented, student-focused, [and] collaborative" (Huang et al., 2019, p. 7). Participants collaborate in a reflective environment with a common goal of improving their students' learning, that also provides important opportunities to develop their practice. Fujii (2018) presents a model of the lesson study cycle with five phases: goal setting, lesson planning, research lesson enacting and observation, post-lesson discussion, and reflection. Sometimes, the participants revise and re-teach the research lesson to a new group of students (Murata, 2011). Several aspects of teaching practice may be promoted in this process, since it involves planning, leading and reflecting.

The lesson study begins by defining the goal of the lesson, which is normally linked to the curriculum. Teachers explore and discuss the curriculum and their current students' difficulties and potentials (Fujii, 2018). Usually, the planning phase is the longest: teachers continue to study considering the goal defined, by exploring tasks, planning in detail, and anticipating students' solutions and difficulties and teachers' interventions. A teacher leads the lesson

planned in her/his classroom, with the other participants observing students' mathematical activity. The empirical evidence collected by the observers is analysed in the post-lesson discussion in which teachers highlight successes and issues concerning student learning. Finally, the participants reflect about the process, and sometimes write a report (Fujii, 2018).

The planned lesson should have a structure and flow that allow access to students' mathematical thinking, being close to a problem-solving lesson (Fujii, 2018; Takahashi, 2021), or an exploratory lesson (Ponte, 2005), which include a whole-class discussion. Stein et al. (2008) list five practice to help teachers to lead productive mathematical discussions: anticipating student responses to the mathematical tasks, monitoring students' responses, selecting responses, sequencing responses, and helping the students make mathematical connections. These practices are very much in line with a problem-solving or exploratory approach, promoting a shift from traditional mathematics education.

Teachers' practice regarding whole-class discussions may be enhanced in a lesson study, but there is a need to better understand how this happens. A way of looking at this is by considering Foucault's (1980) notion of power as productive networks of interventions and activities that produce knowledge, as it can be visible by tracking changes in discourse. The change of each individual discourse can, therefore, provide evidence of a change in teaching practice. To Foucault (1980), power exists in action, that is, power is the expression of practices. If there is no activity, there is no power. Foucault also recognizes that the power is co-constructed by all the participants. In lesson study, all participants are part of this relationship of co-constructing power.

Teachers' discourse about whole-class discussions can be understood as a mode of action (Fairclough, 1992). The idea of intertextuality, present in the analyses of Fairclough and Foucault, stems from the notion that all discourses are anchored in previous ones, since it refers to the condition whereby all communicative events draw on earlier events, with no "free, neutral, and independent enunciation" (Foucault, 1980, p. 114). Thus, teacher's discourse during the whole-class discussion is an intertext that integrates the teacher's prior understanding of whole-class discussions, the involvement in lesson study, and the mathematical activity of their students.

COMMUNICATION IN THE MATHEMATICS CLASSROOM

Communication plays a central role in mathematics teaching and learning, directly influencing students' knowledge development and skill acquisition. Menezes and Nacarato (2020) highlight the importance of

communication in mathematics learning, noting that it takes various forms and is shaped by tasks assigned to students.

Communication between teachers and students is an opportunity to understand students' mathematical activity and to identify teaching practices that can better support learning. A rich communicative environment promotes diverse strategies and negotiation of meanings, moving beyond procedural or correct answers. Guerreiro and Martins (2020) observed a predominance of bidirectional oral interactions between teacher and students. However, they argue that classroom communication should foster challenging conversations about new mathematical ideas and strategies. A whole-class discussion is a privileged phase of the lesson for negotiating and constructing mathematical meaning (Bishop & Goffree, 1986).

This construction emerges from the connection between the ideas under discussion and students' prior knowledge, supported by teachers' questions. Mason (2000) argues that students learn to think mathematically when they are challenged to make their reasoning explicit. Questions, for example, become pedagogical tools that foster engagement and make students' mathematical thinking visible. Franke et al. (2007), contend that teachers should not be expected to provide answers. Instead, they emphasize key elements: repetition, the role of tasks and meaningful questioning.

By encouraging interactions and valuing students' contributions, teachers share mathematical authority and create a dialogic environment (Ponte, 2005). Students should have the opportunity to share their ideas, regardless of correctness. However, whole-class discussions pose challenges, such as maintaining student engagement without turning the lesson into a teacher monologue or fostering a competitive environment where students merely seek to meet perceived expectations. Sociomathematical norms, such as those described by Yackel and Cobb (1996), regulate participation in discussions and help build shared understandings. These norms allow students to develop expectations about learning without feeling pressured to imitate solutions.

For Franke et al. (2007), there are three essential aspects in mathematics teaching and learning related directly to teachers' practice to support productive interactions among students and foster their engagement with new mathematical ideas: (i) creating mathematical discourse; (ii) developing standards that foster mathematical learning; and (iii) creating relationships that support mathematical learning. Also, according to Llinares (2013), teachers' practice involves initiating and guiding interactions, interpreting, and analysing the students' mathematical thinking. Takahashi

(2021) indicates that teachers should not "spend the majority of the class time explaining or demonstrating. However, while the teacher may not talk much during the lesson itself, they must spend lots of time carefully planning the lesson and anticipating students' responses in order to successfully engage them during class" (p. 9).

Planas et al. (2023) notice that many teachers in their study did not create opportunities for students to "participate in the mathematical discourse" (p. 530). They indicate that students only develop their mathematical communication if this also emerges in the dialogue with the teacher. Thus, they consider that mathematics teaching talk is a pedagogic resource to be used by teachers and students. Also from a social constructive perspective, Kooloos (2022) considers that discourse-based teaching, which means a whole-class discourse that builds on students' diverse solutions and ideas (p. 15), is part of teaching practice that develops students' mathematical thinking. As result of his research, Kooloos (2022) highlights this idea of whole-class discourse and whole-class discussion. The first is defined as "conversations about students' various solution methods and ideas", while the second concerns how "students actively engage with each other's ideas" (p. 16).

A whole-class discussion is an important moment for classroom communication. It must be a dynamic and collaborative discussion, facilitated by the teacher and based on the students' solutions. It is a phase of the lesson that seeks to support the exploration of mathematical concepts and processes, enhancing the development of students' mathematical thinking (Faria et al., 2024; Takahashi, 2021).

METHODOLOGY

This study adopts a qualitative and interpretative approach, grounded in the assumption that knowledge is socially constructed, influenced by the context and that it shapes social practice. The first author was a participant observer, assuming the role of facilitator (Lewis, 2016). This involved supporting the flow of sessions, providing resources, and fostering a reflective environment through questioning, without intervening in pedagogical decisions such as task selection or lesson planning. The participants were three teachers from the same school, Marta, Diana, and Petra (pseudonyms), who collaboratively planned, enacted, and reflected on a research lesson focused on angle measurement in grade 5, following Marta and Diana's concerns about students' conceptions. Marta and Diana were participating for the second time in a lesson study, but it was Petra's first experience with this professional development process.

Data was collected through audio recording of the lesson study sessions and through audio and video recording of the research lessons. In total, there were 9 sessions, including the research lessons led by Marta and Diana (Table 1).

Table 1

Lesson study sessions

Session	Duration	Lesson study phases and main activities
1	3 hours	Definition of the research lesson topic: Geometry and measurement.
2 to 5	3 hours each	Definition of the research lesson goal: measurement of angle amplitude. Planning of the research lesson. In collaboration, the group: (a) analyzed the curriculum and several tasks; (b) designed, and applied a diagnostic task, inspired by the work of Bernardo and Rodrigues (2014); (c) adapted a task for the research lesson using students' solutions to the diagnostic task; (d) wrote the lesson plan.
6	1.5 hours	Research lesson led by Diana (joint class arrangement)
		brief post-lesson discussion
7 and 8	50 minutes each	Conclusion of the research lesson in each class separately (Diana and Marta with their classes).
9	3 hours	Post-lesson discussion and reflection

To make it possible for all teachers to observe the research lesson, they decided that it would be led by Diana (session 6), simultaneously for her students and for Marta' students, with a total of 30 students. The students were already used to working in this joint class arrangement. However, since the task and the related whole-class discussion were not completed in this lesson, it was necessary to continue with each class in the next day, this time separately led by Diana in on class and by Marta in the other (sessions 7 and 8)

Data were inductively analysed by discourse analysis, through an articulation of Fairclough's (1992) critical discourse analysis and Foucault's (1980) concept of power. The concept of intertextuality and the concept of power served as analytical lenses to trace teachers' practice, planned or enacted, regarding whole-class discussion. Therefore, we identified discursive changes arising from a new combination of different discourses or noticing the absence of new elements in the discourse (intertextuality) and the interventions and activities that prompted those changes (power as a productive network).

Audios from the data were manually transcribed. From the planning sessions and subsequent post-lesson discussions, excerpts were selected in which the teachers explicitly referred to whole-class discussions, as well as excerpts in which they discussed elements that strongly influence whole-class discussions, such as tasks, time management, diversity of strategies and student questions, and interventions. From the research lessons, excerpts were selected that illustrate the teachers' practice in relation to the aspects discussed during planning. In the post-lesson discussion and reflection, excerpts were selected that directly referred to the whole-class discussion and the aspects mentioned above.

Three discursive marks were inductively identified. These discursive marks made it possible to organize the content of the teachers' discourse into three dimensions, that teachers focused the most when dialoguing and making decisions about whole-class discussions: (i) students' mathematical activity, (ii) lesson structure and interventions, and (iii) lesson study catalysts. Subsequently, the teachers' discourse was tracked throughout all phases of the lesson study in relation to these discursive marks in their relationship with whole-class discussions, based on the notion of intertextuality. Then, based on the notion of power as productive networks, interventions and activities that were found to contribute to leading and reflecting on whole-class discussions were categorised inductively (Table 2).

Table 2

Discursive marks and productive networks identified in teachers' interventions and activities throughout the lesson study.

Discursive marks	Productive networks
(i) Students' mathematical activity, in particular mathematical errors, and misconceptions	**1 Sharing challenges** - Introducing a new discourse by sharing a challenge - Reproducing a discourse by agreeing with a challenge identified by a colleague **2 Facing challenges** - Assuming the need to act on a challenge identified **-** Giving or building on ideas to deal with challenging teaching-learning aspects **3 Sharing knowledge**

(ii) Lesson structure and interventions	- Anticipating students' strategies and difficulties, supported by previous experience or literature - Suggesting tasks or adaptations to tasks **4 Accessing students' mathematical activity** -Analyzing an answer to a diagnostic task, before the research lesson - Analyzing an answer to a research task, after the research lesson - Reflecting on episodes of classroom dialogue **5 Listening actively** - Listening, posing questions and building up on other teachers' interventions during LS sessions - Listening, posing questions and promoting students' interventions during the research lesson **6 Working with tasks** - Exploring/applying tasks - Reflecting on tasks after their application **7 Planning the lesson** - Writing the lesson plan - Planning questions to pose to students - Planning the board - Reflecting on the lesson plan, questions posed and the board after the lesson
(iii) Lesson study' catalysts	**8 Facilitator attitude** **10 Group collaboration** **11 Lesson observation and reflection**

To ensure the credibility and trustworthiness, authors discussed the results, negotiating and validating the interpretations of the data, the discursive marks and the productive networks. Moving between excerpts, theoretical constructs and emerging categories supported a reflexive stance throughout the research process.

RESULTS

Planning the lesson: from fearing students' errors to preparing to use them

Episode 1. For the second session, the facilitator took some tasks (PN6, PN8) to support narrowing down the specific lesson's aim. Petra shared her fear about discussing errors in a whole-class discussion (PN1):

> Petra: *Errors don't always have to be depreciative, but the idea I have is from the old-fashioned teacher. I'm always with that idea of not letting errors show up, I'm always afraid…*
>
> Diana: *It's just like me! I try to make sure it [the error] doesn't stay there!*
>
> Petra: *I'm not going to be a hypocrite… It seems like errors stay there more easily than the correct answer. Therefore, I have a dread of errors.*
>
> Facilitator: *What if there is a discussion around a mathematical error?*

Diana engages in the dialogue sharing the same challenge Petra stated (PN1). This compatibility may have contributed to create a safe environment for sharing what the teachers identified as challenging. It is the facilitator' question that brings a new discourse about mathematical errors in whole-class discussions (PN8). Petra continued by giving more information about her practice when errors emerge during whole-class discussions:

> Petra: *I don't act in the same way because I'm afraid of it. When a person is afraid, she tries not to let it show up too much. Then, what I do is change the tone [of voice] to stay in my other tone. Sometimes I'm explaining something, and that tone is already right. I'm also very visual and audible, that tone is the error tone, so I end up not using that [voice of error] tone afterwards.*
>
> Facilitator: *The change of tone is then a strategy that you end up using in relation to communication and whole-class discussion?*
>
> Petra: *Yes, very much like that.*

The questions from the facilitator fostered Petra to continue sharing her usual practice regarding discussing errors. The facilitator also asked Marta and Diana to reflect and share their view (PN8):

> Facilitator [to Marta and Diana]: *How do you identify with this question?*
>
> Diana: *I'm sometimes afraid that the errors will confuse them, that they will be uncertain between what was called out and what is correct, after it has been talked about and clarified.*

> Marta: *Yes, but there are always some students that, even after being discussed, will maintain the error because they are more absent minded for example. I don't think the same way as Petra and Diana.*
>
> Petra: *But also... I must learn to work with that error because it will happen many times. Oh, my God, isn't it?*

Marta contributed by pointing out that some students may still have questions even after the discussion around an error if they do not pay attention. Therefore, Marta recognized the challenge and seems that see does not agree that avoiding discussing mathematical errors is a solution. The statement from Petra "I must learn to work with that error" is evidence of at least being available to assume the need to act on this challenge (PN2).

Episode 2. By session 4, the teachers had already proposed the diagnostic task in their classes. It had been agreed in advance that, in this session, the group would discuss the mathematical activity of the students in the task in order, if necessary, to adapt the task initially selected for the research lesson (PN3, PN4, PN6):

> Marta: *I think it was useful… This also makes us reflect on our tasks, the representations, for example, that we give in the sheets or in the tests, which sometimes lead them to have some misconceptions that we only realize when we give this type of tasks.*
>
> Petra: *That's true. But why does he paint? He makes the angle here [pointing out to a student solution to the diagnostic task].*

Prompted by the question of the facilitator, Marta began to reflect on the diagnostic task (PN8). She reflected on the importance of representations in the tasks and that these sometimes lead to certain misconceptions, associated with her practice and the impact on students' learning.

Following a question of the facilitator, Petra returned to the analysis of the students' solutions in the diagnostic task, trying to access their mathematical activity (PN2, PN4, PN8). After exploring a few answers from students, the teachers again broadened the reflection to the impact of their practice on students' learning:

> Facilitator: *What do we take away from this? This was a diagnostic task. It's supposed to inform. What do we retain?…*

Petra: *When we start to talk about angles, we always give an acute[angle]. I can't remember using other representations.*

Marta: *We take the kids to certain conditions ourselves.*

Petra: *Instead of opening the possibility for learning, we condition it.*

Diana: *We try to simplify, so that they memorize key things, minimum bases...*

Petra: *Yes, yes!*

Diana: *That sometimes is conditioning.*

Marta: *That's it, we came to that conclusion.*

The interaction between the participants when accessing students' mathematical activity led them to reflect about their practice. Supported by their students' errors and misconceptions, the teachers noticed that when and how they "start to talk" about a content creates an impact on their students' mathematical learning. When it comes to angles, they "always give an acute" angle, which illustrates the lack of diversity and induces the lack of students' interventions during the discussions (PN3).

Petra suggested that in the research lesson, teachers provide students with the solutions to the diagnostic task, which it is quite linked to her fear in session 2: it seems that fearing mathematical errors, Petra's suggestion was to give students a correct solution of the task. However, Diana pointed out that it would be necessary to discuss the students' solutions (PN2, PN4, PN5, PN8):

Diana: *Maybe there were things to clarify here before moving on to measurement. This part [concept of angle] must be well defined.*

Marta: *It seems that there is a missing task to follow up, to be more in line with the questions that they had in this [diagnostic task].*

Petra: *An idea, for example, is to give the correction of this [diagnostic] task.*

Diana: *But this task must be worked on. It must be discussed.*

> Facilitator: *What if we take advantage of this enthusiasm that they [students] have towards this task and use their mathematical activity to prompt a discussion?*
>
> Diana: *And maybe it's a way for them to work on their own results because we would never submit things with errors or must do it wrong.*

After they decided that for the research lesson the diagnostic task would be explored and discussed, the teachers actively listened to each other on how to select the solutions to compare and discuss and the questions to pose (PN5):

> Petra: *So, they can see that their work was really put to good use.*
>
> Marta: *That's going to be a lot of work for the teacher!*
>
> Diana: *We ask them to "justify the choice".*
>
> Marta: *But it's not a justifying process.*
>
> Petra: *Try to explain how your colleague thought to come up with these three answers.*

This dialogue highlights Marta's awareness of the challenge that leading whole-class discussions entails for the teacher (PN1, PN2). The interaction among the teachers to define how to pose the questions to the students is a new discourse appearing from their interactions and also seems to shape Petra's discourse around errors and misconceptions during a whole-class discussion, since she suggested that the students could actively engage with the solutions of their colleagues and to identify that they would discuss their own solutions (PN3, PN5).

Besides the structure of the lesson, Marta promoted a dialogue about the duration for each phase of the lesson and about the importance of selecting the answers to analyse later in the whole-class discussion (PN4, PN5, PN7):

> Marta: *It won't be possible for all the groups to participate because there are eight groups. I think it should be clearly defined in the first discussion which questions it is important to emphasize… Just to select the answers… Maybe things will come up that we don't need now.*

Marta's intervention promoted a new discourse, bringing into the discussion important aspects to plan, such as having the required time so that

students' mathematical ideas are actively heard and discussed. Diana was also conscious of decisions about time management and monitoring student activity, when she said "The class is very big. If after 20 minutes, two or three groups already have conclusions, maybe that's enough to discuss". Petra's interventions are just to agree, reproducing previous discourses, but Marta and Diana's interventions merge new discourses when it comes to deciding lesson structure, time for each phase of the lesson, selecting and sequencing groups for the whole-class discussion. Diana brought to discussion the need to select, among all the solutions from the diagnostic task, those for students to analyse in the research lesson and to discuss.

Leading the whole-class discussion

Episode 3. The research lesson, in session 6, was led by Diana. At the beginning, to introduce the task, both Diana and Marta led the lesson jointly, making interventions to clarify that the students should, in small groups, analyse their classmates' solutions and share their ideas with each other, before the whole-class discussion (PN5, PN6):

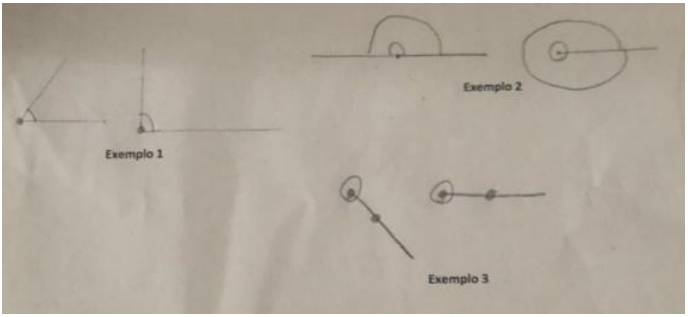
> Diana: *Do you remember the diagnostic task we did? We took that task, selected some answers and we're going to talk and think about it. Each one tries to explain his/her point of view, the other listens and if he/she doesn't agree he/she explains his/her point of view too.*
>
> Marta*: Instead of us [teachers] explaining or correcting, we chose some answers from different students for you to analyse in groups, as teacher Diana said.*

The teachers went beyond encouraging students to identify the solutions as correct or incorrect, by clearly asking them to analyse and discuss possible inaccuracies. Complemented with the fact that the students who authored each solution were not identified in the task, this emphasis on explanation and analysis requested by the teachers fostered a socio-construction of learning and a decrease in the culture of punishing mathematical errors. Therefore, the students' mathematical activity and interventions were fostered in small groups and for the whole-class discussion (PN5).

Episode 4. The discussion continued with Jo's group discussing the second example (Figure 1), negotiating the representations of 360 degrees, 180 degrees and 0 degrees angles:

Figure 1

Notes taken by Jo's group during the autonomous work.



Diana: *We must select.*

Jo: *Well, they both have the dots [intersection points], and they are different angles because they have different amplitudes: the first is a half turn and the second is a full turn.*

Matt: *It can also be an angle with 0 degrees.*

Diana: *Wait. [gives time for Jo to represent the arcs of each angle at the board] So, what do you think [to the class]… The first one is a…*

Class: *Straight angle.*

Diana: *You felt the need to represent the arc in this angle [example 2]. Why?*

Jo: *Yes, to represent the amplitude.*

Diana: *What does amplitude mean?*

Jo: *It is the size of the open space between half-lines.*

In this episode, Matt tried to argue with Jo's group idea, suggesting that one of the angles could be a null angle. Diana questioned Jo about the marking of the arcs in the two angles represented, leading to the emergence of the concept of amplitude, which was defined by Jo as "the size of the open space between rays". So, Diana was listening, posing questions and promoting students' interventions during the research lesson (PN5, PN6). She asked about the second angle represented and Matt again intervened:

Diana: *What about the second one?*

Jo: *It is a full turn.*

Matt: *The second one can also be a 0 degrees angle.*

Diana: *Why?*

Matt: *Because the rays are together.*

Diana: *What is the difference between 0 degrees and 360 degrees? Jo said that it is a full turn, which has 360 degrees, and the rays are juxtaposed. However, Matt said no, that it is a 0-degree angle. Matt, do you agree also with Jo?*

Matt: *Yes, I think it can be both.*

Diana: *Jo, do you agree with Matt?*

Jo: *Hum, maybe.*

Diana: *There is no difference between the representations of those angles?*

Jo: *It is a 0-degree angle when there is no arc, the rays didn't move. A 360-degree angle must have the arc because the rays moved.*

At the beginning of this episode, Diana chooses not to explore Matt's intervention. Opportunely, when Jo shared about the second angle in example 2, Diana asked the difference between the representations of 0 degrees and 360 degrees angles. Even though there was no space for the class to answer this question, Diana returned to Matt and asked if he agreed with Jo about that angle being a full angle. She posed the same question to Jo who, in turn, showed more resistance to agreeing with his colleague, and the negotiation of full and null angle continued. Matt was the student who prompted the discussion about these two angles, and it was possible for students to engage in negotiating the meaning of those representations because Diana's questions allowed then to have access to each other mathematical ideas.

As planned, Diana tried to construct a definition of angle with the participation of the students. Negotiating a definition of angle, as a synthesis, and the students recording in their notebooks took14 minutes. The first part of whole-class discussion exceeded the planned time by 16 minutes, which contributed significantly to time management constrains, not allowing the whole-class discussion to be completed in this lesson.

Episode 5. In session 7, Diana continued the research lesson with her class. The first ten minutes were used by her to recap the definition of angle that students had negotiated. Even though Diana identified during the planning phase the need to follow a lesson structure timeline to guarantee the quality of the whole-class discussion, these two moments illustrate that time management was still challenging (PN7).

Episode 6. In session 8, Marta continued the task with her class. She gave students ten minutes to discuss again in small groups what they had done in the previous lesson and continued the whole-class discussion. She gave each group the opportunity to share their different ideas about question 2 (PN7). After that, she promoted the discussion:

> Marta: *Henry is saying that there is another angle in figure 4.*
>
> Henry: *Yes, concave angles.*
>
> Marta: *Let's use this idea from Henry's group to analyse the second example.*
>
> Student: *The second example is correct, but they didn't need to mark angles inside and outside the figures.*
>
> Marta: *So, you think that there are too many angles. Henry, can you please come here and help us?*
>
> Henry: *We can still mark two more concave angles.*
>
> Marta: *In the beginning, almost all of you were agreeing that there were too many angle marks in the figures. After Henry's example, what do you think?*
>
> Class: *Yes, because we can mark inside and outside the figure*.

By accessing and anchored on students' mathematical ideas, Marta inserted with different colours a legend for the angles that Henry identified as being convex and concave. This intervention brought a discussion regarding the concepts of convex and concave. This is not a powerful mathematical explanation, not even true for all polygons, but these ideas were not addressed in this discussion. In fact, Marta stated during the planning phase that in the whole-class discussion students could come up with mathematical ideas beyond the lesson aim (PN4, PN5).

Episode 7. Question 3 of the task asked students to compare two angles concerning their amplitude. The discussion followed and Peter concluded that

the amplitude of an angle does not depend on the length of the rays, therefore, it is possible to compare angle amplitudes without a protractor if both angles have a horizontal arm.

> Marta: *In question 3, Ali said that he would need a protractor to measure each amplitude. What do you think, Peter?*
>
> Peter: *The sides of the angle are rays, So, if we continue to mark them…*
>
> Marta: *Peter, show us that on the board.*

Peter went to the board and extended the rays of the angle. One of the misconceptions identified with the diagnostic task was that most of these students though the angle was a finite area, therefore, it was important to discuss the concept of ray linked to the concept of angle (PN4, PN6).

Reflecting on the lesson

Episode 8. In the reflection session, Petra recalled her fear about discussing errors in a whole-class discussion, stating that she believed, in the beginning of the lesson study, that "I was afraid of letting errors show up". From her side, Diana reflected about the contribution of applying diagnostic tasks, which enable teachers to have a deeper understanding of the possible misconceptions or ineffective strategies students may use during the research lesson to first address the aim defined. Diana considered diagnostic tasks a good instrument to access students' mathematical activity before the research lesson (PN1, PN2, PN4):

> Diana: *I think it was in line with what we had anticipated might happen, but perhaps we were at an advantage because we had what they had done in the diagnostic task. So, we ended up anticipating more easily or we had more anticipation.*

Episode 9. Diana, now about the type of task used in this research lesson (PN5, PN6, PN7), stated that:

> Diana: *The task used was challenging because the students needed to write their justifications. In the future, I think we must use this type of task, where they can evaluate given solutions, some of them with errors or wrong ideas. Although it's difficult for us to lead whole-class discussions and manage time with this type of task, it's exciting to realize that the students are involved.*

In this intervention, Diana values working with a different type of task to promote students' engagement in the discussion, even to identify their errors and misconceptions. Still, she recognizes how challenging it can be to lead a whole-class discussion with such tasks and to manage time. However, she shows an intention of continuing to use these tasks since it is motivating for her to notice students being involved in.

Episode 10. In this lesson study, when reflecting on the research lesson, Marta considered that students were actively listening to their peers, which is also a significant illustration of the teachers being able to listen and promote students' interventions (PN5). In Marta' reflection, time management appears to be a challenge that teachers need to face, and she assumed the need to act on this challenge (PN2, PN7):

> Marta: *I think that the moment of the whole-class discussion went very well. I noticed that the students listened to each other and tried to make their own contributions. They learned. The most challenging part for us is the management of time, because we don't want to cut off the students' participation either ... We must give them a set amount of time to finish the task, which is for them to improve their work pace... And then, maybe, instead of us repeating things, we should continue the lesson.*

Regarding the group collaboration and activities done, Marta emphasized a different way of collaborating with her colleagues (PN9, PN10).:

> Marta: *There was mutual help before, during and after the lesson and real collaboration. Even though I was conducting the lesson, I felt that it was the result of everyone's work, which gave me greater comfort. The collaborative work carried out in the sessions proved to be more fruitful than what is usually done in school, in which we just share materials that have already been developed and adapt them to our classes.*

These reflections mainly illustrate that identifying challenges and facing challenges were very productive in the lesson study. These productive networks seem to be associated with the fact that a diagnostic task was undertaken in this lesson study and, as a result, the task for the research lesson was built from the students' solutions with errors and misconceptions identified in the task. The group collaboration consisted of mutual help, which they also identified as powerful.

DISCUSSION

Productive networks

During this phases most teachers' interventions include at the same time two of the discursive marks identified. Episode 2, for example, shows that the students' mathematical activity and the lesson structure and interventions are largely related, not only during the lesson but also in planning, when Petra shares that she always introduces students to the concept of angle with the representation of an acute angle (teacher intervention), which may lead to the development of misconceptions (students' mathematical activity). In episode 3, for example, teachers make it clear to the students that there will be a phase of small group work and then a whole-class discussion (lesson structure and interventions), inviting students to analyse and talk about their peers' solutions (students' mathematical activity).

The third discursive mark, lesson study' catalysts, has two productive networks (PN9, PN10) that appear in the reflection. Still, the facilitator' attitude (PN8) is identified since the planning phase. Table 3 synthesizes the distribution of these networks across episodes, allowing us to trace relationships between productive networks and lesson study phases.

Table 3

Productive networks identified in the planning phase, research lesson and post-lesson discussion and reflection

Planning phase	Research lesson	Post-lesson discussion and reflection
Episode 1 – PN1, PN2, PN6, PN8 Episode 2 – PN1, PN2, PN3, PN4, PN5, PN6, PN7, PN8	Episode 3 – PN5, PN6 Episode 4 – PN5, PN6 Episode 5 – PN7 Episode 6 – PN4, PN5, PN7 Episode 7 – PN4, PN6	Episode 8 – PN1, PN2, PN4 Episode 9 – PN5, PN6, PN7 Episode 10 – PN2, PN5, PN7, PN9, PN10

Although sharing knowledge (PN3) appears explicitly only during the planning phase, Diana later emphasized in the post-lesson discussion the importance of having accessed students' mathematical activity through a diagnostic task (PN4), sharing knowledge, particularly anticipating strategies based on prior experience or literature, and suggesting modifications to a task, was closely linked to accessing students' mathematical activity. This process of identifying common errors and misconceptions related to angle learning

enabled Diana to feel more confident in the whole-class discussion. This is evidence of the importance of planning in detail, specifically the whole-class discussion (Stein et al, 2008; Faria et al., 2024) so that teachers can feel more confident and prepared.

During the enactment of the lesson, the most visible productive networks were listening actively (PN5) and working with tasks (PN6). These two networks are aligned with the promotion of meaningful students' participation (Bussi, 1991; Brendefur & Frykholm, 2000; Henning et al., 2012). The productive network of accessing students' mathematical activity (PN4) was indirectly but strongly present during the whole-class discussion, particularly through the questions posed by teachers (Herbel-Eisenmann & Breyfogle, 2005; Llinares, 2013). These questions showed that they had considered the previously identified errors and misconceptions, guiding the discussion to foster students' thinking around those ideas.

Although the productive network planning the lesson (PN7) did not involve a specific action during the enactment phase, it became clear when, for example, Diana faces challenges with time management, which compromised the execution of the previously structured lesson plan. Sharing and facing challenges (PN1, PN2), as well as the facilitator's attitude (PN8), are networks that appeared mostly during the planning phase. However, the fact that the teachers led a discussion enriched by students' errors and misconceptions suggests that the research lesson represented a meaningful challenge they faced. Therefore, there is evidence to support Takahashi (2021) statement regarding teachers needing support to deal with challenges and try different approaches. Next, we explore each of the three phases of lesson study.

Planning the lesson

Reflecting on discussing errors and misconceptions in whole-class discussion was particularly noteworthy during the planning. Although Diana and Petra took a similar approach when this challenge emerged, Marta's interventions and the facilitator's questions promoted a safe environment for sharing, and also for reflecting on possible changes in their teaching practice.

While planning, teachers made decisions linked, directly or indirectly, to the whole-class discussion. Diana, for example, suggested to stop the autonomous work on time, even if no groups had finished the task, ensuring time for the whole-class discussion, a very important aspect to consider (Faria et al., 2024). Petra showed she was available to face her fears of discussing errors, specifically when she linked the selection of different solutions to

students' feeling valued. As it is important to anchor the whole-class discussion on students' solutions (Ponte & Serrazina, 2004; Stein et al. 2008), Marta focused on selecting groups to share their ideas and defining clearly what to emphasize and synthesize concerning the lesson aim and to anchor on students' mathematical activity. While considering time management for a whole-class discussion and the need to anchor it on students' solutions, Petra, Diana and Marta were quite aligned to the idea of Takahashi (2021) that teachers need to help students engage in the discussion and lead them to help each other. Table 4 provides a synthesis of teachers' interventions that illustrate how their discourse about the whole-class discussion changed between sessions 2 and 4.

Table 4

Changes in teachers' discourse on whole-class discussions during the planning phase,

Session 2 (Episode 1)	**Session 4 (Episode 2)**
Petra is afraid to discuss errors in the whole-class discussion and Diana tends to agree with Petra; Marta is not totally aligned with Petra and Diana; Facilitator encourages discussion among teachers.	Shift from avoidance to exploration of errors and misconceptions; Recognition of discussing errors as learning opportunities; Discussion about tasks features or teacher's interventions may enable students to build misconceptions; Recognition of the need to discuss tasks with students' participation in whole-class discussion, instead of only providing right solutions; Recognition of the importance of selecting solutions and manage time for the whole-class discussion; Facilitator supports task analysis and planning.
Productive network: 1, 2, 6 and 8	**Productive networks:** 1, 2, 3, 4, 5, 6, 7, 8

In session 4, the group dialog considered crucial aspects that highly influenced the whole-class discussion, as the influence of tasks, teachers' interventions and students' difficulties. After the interventions of Marta and the facilitator, Petra, showed an intended change in her practice, acknowledging that she must learn to work with errors because they will happen many times. This aligns with research showing that discussing students' mathematical errors and misconceptions improves the quality of a shared discourse between teachers and students, promoting an interactive construction of students' mathematical meanings (Ponte & Serrazina, 2004; Takahashi, 2021).

The questions posed by the facilitator led teachers to discuss. Therefore, the facilitator's interventions contributed to emerging other productive networks, namely sharing/facing challenges and accessing students' mathematical activity. For example, in episode 1 the facilitator challenged teachers to think about whole-class discussions as an appropriate occasion to discuss errors and misconceptions and to clarify Petra's strategies for managing communication during the whole-class discussion. In episode 2, the facilitator's questions sought to access teachers' perceptions, namely about the task and students' knowledge. These questions contributed to teachers develop broader dialogues, namely about the relationship between their practice and students' learning (Lewis, 2016).

Even though the teachers had already read about angle misconceptions (Bernardo & Rodrigues, 2014) and had already selected a task for the research lesson, in session 4, after exploring and reflecting on students' solutions to the diagnostic task they reconsidered their decision. The task for the research lesson should now address students' errors and misconceptions regarding the concept, before measuring and exploring other angle related notions. This key moment made the group consider that students would explore their own solutions to the diagnostic task in small groups and later discuss them in whole-class. Regarding the aim of the lesson, its planned and the final task, the whole-class discussion was prepared to address misconceptions about the angles through students' interventions, having as starting point their solutions to the diagnostic task. As Takahashi (2021) indicates, addressing misconceptions through debate is a type of discussion with potential for students' learning.

Studying the content, exploring what research says about learning that content and anticipating students' strategies and difficulties are usual activities in planning in a lesson study. Although the teachers had read about misconceptions, it was from the students' diagnostic task that the teachers realized that their practice induced these misconceptions. Therefore, the result of the diagnostic task had a deep impact on the teachers' decision-making, namely in what aspects of the content to discuss and how.

Research lesson

In the research lesson, teachers encouraged students to explore the different solutions present in the task. Marta and Diana prompted the discussion in small groups, which is a way of preparing the whole-class discussion. As debating over correct and incorrect answers may engage students in authentic discussion, the teachers created an environment where students could negotiate

mathematical meanings such as angle, ray and amplitude, instead of just reproducing definitions presented by the teacher.

As the interactions among the students and between the teacher and the students during the whole-class discussion determine its quality (Boerst et al., 2011), Diana tried to create opportunities for discussion among students about the concept and the representation of different angles. Her interventions encouraged students to listen, analyse and respond to their colleagues' reasoning, as well as to argue and constructively criticize their interventions. Marta also promoted interactions among the students, namely regarding the misconception that the angle is a finite area. Even though to identify convex and concave angles was not part of the aim of the lesson, the students brought this for discussion. These concepts were not explored in this lesson, but having students communicate about them was a springboard for further exploration.

During the whole-class discussions, Diana and Marta's interventions were mostly questions, creating opportunities for students to engage in the discussion (Kooloos et al., 2022). Although the dialog between the students during the discussion did not take the form of questioning each other, there were moments of disagreement and negotiation of meanings, driven by questions posed by the teachers (Herbel-Eisenmann & Breyfogle, 2005).

In the research lesson, teachers accessed students' mathematical activity by the moment they were working in small groups, before the discussion, allowing them to monitor students' mathematical activity (Stein et al., 2008). Still, during the discussion, teachers asked new questions that were important for achieving the lesson's aim. For example, Diana asked the meaning of amplitude and the difference between the representation of angles of 0, 180 and 360 degrees, and Marta valued a student intervention about convex and concave angles, although she filtered out these, as they were not part of the lesson aim.

Post-lesson discussion and reflection

During the post-lesson discussion and reflection, teachers recognized that they had identified and collaboratively addressed challenges. As a result, Diana associated the process of confronting these challenges with a deepening of students' access to mathematical activity, not only through a diagnostic task, but also through a thorough analysis of students' solutions to that task, which enabled a more detailed anticipation during the planning. This aligns with Fairclough's (1992) notion of intertextuality, as Diana's discourse was shaped by the formative social context of the lesson study. It also resonates with

Foucault's (1980) concept of productive network, in the sense that the interactions and activities in which Diana engaged contributed to the development of her knowledge about students' mathematical activity and her teaching practice.

Active listening and lesson planning were productive networks revisited by the teachers in connection with the type of task used. Diana noted that, although managing time (PN7) is difficult when using such tasks (PN6), it is more motivating because she perceives that students are engaged (PN5). It appears that placing students at the centre of learning, as expected in a lesson study (Fujii, 2018; Takahashi, 2021), was achieved by Diana, an aspect that highly influence how teachers promote students' interventions. Similarly, Marta also emphasized in her reflection the students' engagement, which she partially attributed to her active listening and to the nature of the task chosen. Nevertheless, Marta also acknowledged the challenge associated with time management (PN7).

Collaboration and lesson observation are two productive networks that are clearly identified by Marta as contributing to teachers' practice in this lesson study. Marta stated that she thoughted that she already collaborated with colleagues in her school, but lesson study prompted a more effective collaboration and reflection. This aligns with Richit and Ponte (2017), showing that lesson study in a process that can put teachers to question their previous ideas, and consequently transform them.

CONCLUSIONS

This study sought to understand how lesson study may contribute to teachers' practice regarding whole-class discussion by identifying what productive networks link teachers' planning to lead and reflect on whole-class discussion. We recognize three discursive marks: (i) students' mathematical activity, especially mathematical errors and misconceptions; (ii) lesson structure and interventions, and (iii) lesson study's catalysts. The first two are associated with these productive networks: sharing challenges, facing challenges, sharing knowledge, accessing students' mathematical activity, actively listening, working with tasks and planning the lesson. The third mark is associated with facilitator attitude, group collaboration and the observation/reflection of the research lesson.

This study illustrates how lesson study may promote productive networks that link planning to lead and reflect on whole-class discussions. During planning, we identified critical moments in which teachers reproduce

discourses from each other (e.g., sharing challenges) or faced challenges by producing new discourses (Jorgensen & Philips, 2002), sometimes with the contribution of the facilitator. The collaborative environment among teachers was fuelled by these interventions and lesson study's activities, like enacting and exploring a diagnostic task, which provided access to students' mathematical activity before the whole-class discussion.

These productive networks do not seem to be attached to unbalanced power relations or to a continued overlapping of voices. For instance, Marta's interventions were highly powerful in promoting new discourses about the whole-class discussion, namely about time management and selecting solutions to discuss. Time management was a challenge, especially for Diana. She posed purposeful questions, sequenced solutions, and anchored the syntheses on students' contributions. However, she revoiced the same ideas several times during the synthesis even though she was very aware of this issue in her practice. Despite time constraints, both Marta and Diana went much beyond unidirectional communication (Brendefur & Frykholm, 2000). The nature of the task and the teachers' interventions during the whole-class discussion prompted the students to actively engage with each other's ideas, instead of just having conversations about their solutions (Kooloos, 2022).

Productive networks identified during the planning, even if not directly observable during the research lesson or post-lesson reflection, are inherently interconnected with other networks. Sharing challenges, facing challenges, and accessing students' mathematical activity through a diagnostic task were especially relevant. Overall, productive networks did not operate in isolation; rather, they were interrelated and mutually reinforcing, which aligns with Fairclough's (1992) notion of discourses as being reshaped across social practices. It also echoes the view of lesson study as a process of teacher professional development illustrating changes between teachers planned practice and their enacted practice.

A limitation of this study is that Petra did not have opportunity to teach a research lesson and lead a whole-class discussion. Although we identify changes in her discourse along the planning, observing her practice would enrich the results. Further investigations should also consider how these productive networks are related to the presence of an external facilitator and how they may evolve considering teachers' further participation in lesson study.

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AUTHORS' CONTRIBUTIONS STATEMENTS

FF collected and made a preliminary analysis of the data, wrote the first version of the theoretical framework and the research methodology. FF, JPP, MR and MQ designed and discussed all parts of the article, reviewed and contributed to the final version of the manuscript. All authors approve final version of the article.

DATA AVAILABILITY STATEMENT

The corresponding author, FF, will make the data supporting the results available upon adequately justified request.

REFERENCES

Bernardo, M. I., & Rodrigues, M. (2014). Desenvolvimento do conceito de ângulo: Um estudo no 5. ºano de escolaridade. *CIED – Centro Interdisciplinar de Estudos Educacionai*s. Escola Superior de Educação de Lisboa. https://hdl.handle.net/10400.21/3559

Bishop, A. J., & Goffree, F. (1986). Dinâmica e Organização da Sala de Aula. In B. Christiansen, A. G. Howson & M. Otte (Eds.), Perspectives on Mathematics Education, 309–365.

Boerst, T., Sleep, L., Ball, D. & Bass, H.P. (2011). Preparing teachers to lead mathematics discussions. *Teachers' college record*, *113*(12), 2844-2877. https://doi.org/10.1177/016146811111301207

Brendefur, J., & Frykholm, J. (2000). Promoting mathematical communication in the classroom: Two preservice teachers' conceptions and practices. *Journal of Mathematics Teacher Education, 3*(2), 125–153. https://doi.org/10.1023/A:1009947032694

Bussi, M.B. (1991). Social interaction and mathematical knowledge. *Proceedings of the 15th International Conference for the Psychology of Mathematics Education*, 1- 16. Assis, Itália.

Fairclough, N. (1992). *Discurso e mudança social* (Trad. I. Magalhães). Editora da Universidade de Brasília.

Faria, F. , Ponte, J.P., & Rodrigues, M. (2024). Teachers' leading whole-class discussions in a mathematics lesson study: From initial understanding to orchestration in practice. *European Journal of Science and Mathematics Education*, 12(1), 1-18. https://doi.org/10.30935/scimath/xxxx

Foucault, M. (1980). Truth and power. In C. Gordon (Ed.) *Power/knowledge: Selected interviews and other writings 1972–1977*. Harvester.

Franke, M.L., Kazemi, E., Battey, D. (2007). Understanding teaching and classroom practice in mathematics. In F. Lester (Ed.), *Second handbook of mathematics teaching and learning* (pp. 225-256). Information Age.

Fujii, T. (2018). Lesson study and teaching mathematics through problem solving: The two wheels of a cart. In M. Quaresma, C. Winsløw, S. Clivaz, J.P. Ponte, N. Shuilleabhain & A. Takahashi (Eds.), *Mathematics lesson study around the world* (pp. 1–21). Springer. https://doi.org/10.1007/978-3-319-75696-7_1

Guerreiro, A., & Martins, C. (2020). Avaliação e comunicação na aula de matemática: conceções e práticas de professores do 2.º ciclo do ensino básico. Quadrante, 29(2), 6–23. https://doi.org/10.48489/quadrante.22570

Henning, J.E., McKeny, T., Foley, G.D., Balong, M. (2012). Mathematics discussions by design: Creating opportunities for purposeful participation. *Journal of Mathematics Teacher Education*, 15, 453-479

Herbel-Eisenmann, Beth A., M. Lynn Breyfogle. (2005). Questioning Our Patterns of Questioning. *Mathematics Teaching in the Middle School* 10, no. 9, 484–89. https://doi.org/10.5951/MTMS.10.9.0484

Huang, R., Takahashi, A., da Ponte, J.P. (2019). Theory and Practice of Lesson Study in Mathematics around the World. In: Huang, R., Takahashi, A., Ponte, J.P. (Eds) Theory and Practice of Lesson Study in Mathematics (pp. 3-12). Advances in Mathematics Education. Springer, Cham. https://doi.org/10.1007/978-3-030-04031-4_1

Jorgensen, M., & Phillips, L. J. (2002). *Discourse analysis as theory and method*. SAGE. https://doi.org/10.4135/9781849208871

Kooloos, C. C. (2022). *Eye on variety: The teacher's work in developing mathematical whole-class discussions* [PhD thesis, Radboud Universiteit Nijmegen].

Lewis, J.M. (2016). Learning to lead, leading to learn: How facilitators learn to lead lesson study. ZDM, 48(4), 527–540. https://doi.org/10.1007/s11858-015-0753-9

Llinares, S. (2013). Professional noticing: A component of the mathematics teacher's professional practice. *Sisyphus Journal of Education*, 1(3), 76-93.

Mason, J. (2000) Asking mathematical questions mathematically. International Journal of Mathematical Education in Science and Technology, 31:1, 97-111, https://doi.org/10.1080/002073900287426

Menezes, L., & Nacarato, A. M. (2020). Comunicação no ensino e na aprendizagem da Matemática. Quadrante, 29(2), 1–5. https://doi.org/10.48489/quadrante.22568

Murata, A. (2011). Introduction: Conceptual overview of lesson study. In L. C. Hart, A. S. Alston & A. Murata (Eds.), Lesson Study Research and Practice in Mathematics Education (pp. 13–24). Springer.

Planas, N., Adler, J., & Mwadzaangati, L. (2023). What is mathematics teaching talk for? A response based on three sites of practice in

mathematics education. *ZDM Mathematics Education, 55*(3), 521-534.

Ponte, J. P. (2005). Gestão curricular em matemática. In GTI (Ed.), *O professor e o desenvolvimento curricular* (pp. 11-34). APM.

Ponte, J.P., & Serrazina, L. (2004). Práticas profissionais dos professores de Matemática. Quadrante, 13(2), 51-74.

Ponte, J.P., Branco, N., Quaresma, M., Velez, I., & Mata-Pereira, J. (2012). Perspetivas teóricas no estudo das práticas profissionais dos professores de matemática. In A. P. Canavarro, L. Santos, A.M. Boavida, H. Oliveira, L. Menezes & S. Carreira (Eds.), *Práticas de ensino da Matemática: Atas do Encontro de Investigação em Educação Matemática* (pp. 267-279), SPIEM.

Richit, A., & Ponte, J. P. (2017). Teachers' perspectives about lesson study. *Acta Scientiae*, 19(1), 20-30.

Stein, M.K., Engle, R.A., Smith, M., & Hughes, E.K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, 10, 313-340. https://doi.org/10.1080/10986060802229675

Stigler, J.W. & Hiebert, J. (2019). Preface. In R. Huang, A. Takahashi, & J.P. Ponte (Eds.), *Theory and practice of lesson study in mathematics: An international perspective*. Springer.

Takahashi, A. (2021). Teaching mathematics through problem-solving: A pedagogical approach from Japan. Routledge. https://doi.org/10.4324/978100301547

Yackel, E. & Cobb, P. (1996). Normas sociomatemáticas, argumentação e autonomia em matemática. Journal for Research in Mathematics Education, 27(4), 458-477.