

Becoming a mathematics teacher educator: Identifying Characteristics of learning opportunities

Alessandro Ribeiro^a ORCID iD (<https://orcid.org/0000-0001-9647-0274>)

Salvador Llinares^b ORCID iD (<https://orcid.org/0000-0002-0801-316X>)

^a Federal University of ABC, Center of Mathematics, Computation and Cognition, Santo André, Brazil

^b University of Alicante, Faculty of Education, San Vicent del Respeig, Spain

ABSTRACT

Background: This paper aims to characterize learning opportunities of novice mathematics teacher educators (MTEs) when they plan, facilitate, and reflect on a practice-based professional development program for secondary mathematics teachers. **Objectives:** The aim of the study is to characterize novice mathematics teacher educators' learning opportunities when they design and implement a practice-based professional development for mathematics secondary teachers. **Design:** This is a qualitative-interpretative study seeking to understand the subjective experiences and perspectives of the novel mathematics teacher educators as well as uncover the meanings that participants attribute to their lives and experiences. **Setting and Participants:** The study was developed in a practice-based development course involving three novice MTEs and an expert MTE with diverse academic and professional backgrounds. **Data collection and analysis:** Data are the transcriptions of ten video-recorded planning sessions (sessions aimed at designing and reflecting on the enactment), and we carried out a systematic content analysis involving data organization, coding, theme development, and reflexivity. **Results:** Findings illustrate three features of the interactive settings defining MTEs' learning opportunities: (i) Problematizing what has been assumed; (ii) Seeking a balance between cognitive, social, and affective aspects; and (iii) Modelling the expert MTE's role as a broker. **Conclusions:** The dynamic interplay of these features defines learning opportunities for the novice MTEs, conveying the idea of learning defined by different, changing aspects over time, constantly being shaped by its diverse components that contribute in varying ways at different times.

Keywords: Mathematics teacher educator; Sociocultural perspective of learning; Professional development of mathematics teachers; Critical events.

Tornando-se um formador de professores de matemática: identificando características de oportunidades de aprendizagem

Corresponding author: Alessandro Ribeiro. Email: alessandro.ribeiro@ufabc.edu.br

RESUMO

Contexto: Este artigo tem como objetivo caracterizar as oportunidades de aprendizagem de educadores de professores de matemática iniciantes (FPMs) quando planejam, facilitam e refletem sobre um programa de desenvolvimento profissional baseado na prática para professores de matemática do ensino médio. **Objetivos:** O objetivo do estudo é caracterizar as oportunidades de aprendizagem de formadores de professores de matemática iniciantes quando eles projetam e implementam um desenvolvimento profissional baseado na prática para professores de matemática do ensino médio. **Design:** Este é um estudo qualitativo-interpretativo que busca compreender as experiências e perspectivas subjetivas dos novos formadores de professores de matemática, bem como descobrir os significados que os participantes atribuem às suas vidas e experiências. **Ambiente e participantes:** O estudo foi desenvolvido em um curso de desenvolvimento baseado na prática envolvendo três FPMs novatos e um FPM especialista com diversas formações acadêmicas e profissionais. **Coleta e análise de dados:** Os dados são as transcrições de dez sessões de planejamento gravadas em vídeo (sessões que visam projetar e refletir sobre a promulgação) e realizamos uma análise sistemática de conteúdo envolvendo organização de dados, codificação, desenvolvimento de tema e reflexividade. **Resultados:** Os resultados ilustram três características dos cenários interativos que definem as oportunidades de aprendizagem dos FPMs: (i) Problematicar o que foi assumido; (ii) Buscar um equilíbrio entre os aspectos cognitivos, sociais e afetivos; e (iii) Modelar o papel do MTE especialista como um *broker*. **Conclusões:** A interação dinâmica desses recursos define oportunidades de aprendizagem para os FPMs novatos, transmitindo a ideia de aprendizagem definida por aspectos diferentes e mutáveis ao longo do tempo, sendo constantemente moldada por seus diversos componentes que contribuem de maneiras variadas em momentos diferentes.

Palavras-chave: Formador de professores de matemática; Perspectiva sociocultural da aprendizagem; Desenvolvimento profissional de professores de matemática; Eventos críticos.

INTRODUCTION

Teacher educators play a fundamental role in improving the quality of education (Goodwin & Kosnik, 2013), which has led in recent years to growing research, whether in general (Cochran-Smith & Zeichner, 2005) or on those working in the areas of science or mathematics (Goos & Beswick, 2021; Jaworski, 2008; Krainer et al., 2021). This situation has generated the need to understand better how novice mathematics teacher educators learn to do their work (Schwartz et al., 2021). In particular, those who work with secondary mathematics teachers because the specificities of their knowledge and practices (Wasserman et al., 2023). Mathematics teacher educators (MTEs) include “anyone engaged in the education or development of teachers of

mathematics” (Beswick & Goos, 2018, p. 418). We understand a mathematics teacher educator (MTE) as a professional who educates prospective mathematics teachers or facilitates and supports the professional development of teachers already in practice (Krainer & Llinares, 2010). So far, the knowledge and practices of mathematics teacher educators have been studied, but less is known about how they become MTEs, and this seems a consequence of the scarcity of formal spaces to become one (Even & Krainer, 2014; Wu et al., 2020) and of the lack of conceptual frameworks to explain their learning (Superfine et al., 2024; McDonald et al., 2013; Forzani, 2014). Currently, we need to understand better how mathematics teacher educators learn to do their work and identify the characteristics of the context in which this learning occurs.

Although for several years, questions have been raised regarding the MTEs’ learning of new practices and how they can acquire expertise (Goos, 2020; Knight et al., 2014; Krainer et al., 2021; Olanoff et al., 2021; Ping et al., 2018; Schwarts et al., 2021), our knowledge about the setting in which this learning happened is still scarce. Over the last few years, diversity of approaches to teacher educators’ education have emerged, considering different settings (initial teacher education and professional development programs) (Chorney et al, 2025) and different focus such as the processes of disciplinary boundary crossing and identity transformation in context of collaboration between mathematics teacher educators with different background (Ozmatar & Agaç, 2025) as they transition from secondary mathematics teacher to secondary teacher leader. However, since teachers and teacher educators with different background profiles are becoming increasingly involved in the work of teacher educators, we need new knowledge about what factors define learning opportunities. Furthermore, little is still known about the learning experiences of mathematics teacher educators with diverse backgrounds in collaborative environments (Borko et al., 2014; Loughran, 2014; Ribeiro & Ponte, 2019; Ribeiro & Ponte, 2020; Superfine & Pitvorec, 2021; Doná & Ribeiro, 2024). In particular, we need to understand the learning of MTEs when they are immersed in a collaborative work such as a community of practice (Wenger, 1998), understood as a space that enables learning and professional development of participants (Goos & Bennison, 2018; Olanoff et al., 2021), specifically, when they are involved in approaches centered on the direct enactment of high-leverage teaching practices.

Building on previous studies, our work aims to contribute new insights into how novice secondary mathematics teacher educators acquire the

skills necessary for their work and how contextual factors influence their learning and development (Knight et al., 2014; Chen et al., 2018). In particular, when novice mathematics teacher educators with different expertise profiles – e.g., mathematicians and mathematics teacher educators– design, facilitate, and reflect on a professional development program addressed to secondary mathematics teachers. This focus takes into account how MTEs’ participation in a practice-based intervention can provide different learning opportunities for them (Chapman, 2021; Knight et al., 2014), as well as how novice mathematics teacher educators have opportunities to construct shared knowledge (Jaworski, 2008). A learning opportunity refers to any situation that enables novice mathematics teachers to acquire new ways of thinking that enhance their professional practice when they collaborate in planning professional development, reflecting on facilitation, and solving common challenges. With this research problem in mind, we assume two premises that underpin our study: (i) we adopt a practice perspective of learning and professional development of a MTE (Goos, 2020); and (ii) we explore ways of representing MTEs’ knowledge “as a complex system or way of thinking” providing the support for a particular practice (Chapman, 2021, p. 412).

Our research goal is to identify the characteristics of learning opportunities for novice mathematics teacher educators with diverse backgrounds participating in a practice-based professional development program for in-service secondary mathematics teachers.

THEORETICAL FRAMEWORK

We adopted a sociocultural perspective on mathematics teacher educators’ learning as a move from being a novice MTE to an experienced MTE through interactions with others, developing forms of collaborative analysis and interpretation. This perspective emphasizes the role of social interaction, cultural tools, and contextual influences in shaping how novice mathematics teacher educators learn a new practice. We situate the MTEs’ learning into collaboration and dialogue with other novice MTEs, expert MTEs, and practicing secondary teachers by participating in the design and facilitating a professional development project. This perspective assumes that MTEs learn by enacting new practices through engaging with the design of professional tasks, facilitating professional development for secondary mathematics teachers, and reflecting on what has been done (Zaslavsky, 2008).

We investigated MTEs’ learning in terms of the ways they participate when designing professional development for in-service secondary

mathematics teachers. We consider this approach an instance of practice-based teacher educator learning, as MTEs learn how to cope with the demands of facilitating professional development, thereby defining a context in which the transition from their personal background profiles to expertise as MTEs may occur. Three notions are relevant in this transition: the notion of artifact, the meanings of participation and pedagogical reasoning, and the notion of context mediating the MTEs' learning.

First, an artifact is understood as a cultural tool or object that people use to mediate their interactions, thereby influencing their learning. When we studied the novice MTEs' learning, the interactions generated a shared practice, such as designing professional learning tasks (PLTs as artifacts) for use in professional development, their implementation, and reflection on what happened (collegial reflection). We considered the design of professional learning tasks (PLTs) and the discussions about how to enact them as a practice that mediates the learning of novice MTEs, influencing how they think and act. We assumed that a PLT plays the role of a cultural tool that mediates learning.

Second, we assumed that pedagogical reasoning (Loughram, 2019), underpinning the decision-making, actions, and intents of teacher educators, originates in social interaction, and these cognitive processes are subsequently internalized. From this perspective, teacher educators construct knowledge by interacting with others in shared activities, such as designing, implementing, and reflecting on the implementation of practice-based professional development. In this setting, meaning is negotiated through these interactions.

Third, the setting is the design and implementation of a practice-based professional development for practicing secondary mathematics teachers. From a sociocultural perspective, novice MTEs' learning involves interpreting and reshaping their knowledge through the practices of designing professional learning tasks (the artifacts) and ways of using them, as well as discussing the challenges that arise when enacting specific practices through participation. In our study, a PLT is formed by two representations of practice. First, a set formed of mathematical problems in secondary school, questions aimed at analyzing its cognitive demands and anticipating students' difficulties, as well as ways of launching and managing the problems in secondary classrooms. Second, a set of secondary students' answers to the mathematical problem illustrates several features of students' mathematical learning and issues about how teachers could interpret them.

In a practice-based teacher education, teaching practice assumes a central role in the teacher education processes (Ball & Cohen, 1999) and the intervention focuses on developing teachers' skills to enact core teaching practices such as noticing teaching situations (mathematical tasks, students' mathematical thinking, teachers' moves and so). Furthermore, it underlines the ability to allow teachers to engage with experiences in spaces of collective work and discussion that encourage reflection on their knowledge and the sharing of their experiences in classroom practice (Ball & Forzani, 2009). Such aspects need to be provided by teacher educators who, in turn, also need to be involved in the design and implementation of this practice-based learning approach.

The design of a professional learning task helps mathematics teacher educators to collaborate and learn despite having different perspectives. Designing PLTs may help novice MTEs from different backgrounds work together (defining a joint enterprise) and support joint decision-making. The PLTs are collaboratively developed by the MTEs and are revised in conversations about how the implementation will happen. We highlighted the value of collaborative inquiry around PLT as artifacts of practice, serving as a learning tool for MTEs, and considering new practices that differ from MTEs' prior work.

This approach underlines collegial conversations around artifacts as an effective way to support MTE learning and development (van Zoest & Levin, 2021). Furthermore, in the conversations about how the implementation happened, the decision-making processes may reveal features in how the novice mathematics teacher educators' background expertise profiles determine their participation and the negotiation process when a common goal for collegial inquiry is established (in this case, designing a PLT in a practice-based professional development program for in-service secondary mathematics teachers). So, a PLT acts as a mediator (intermediary) between MTEs and their environment; hence, we can assume that the design of PLT shapes how MTEs think, communicate, and act, filtering their ways of participating and interacting through a process of meaning negotiation. These settings can generate learning opportunities for MTEs (Chen et al., 2018; Chorney, S., et al. 2025; Karsenty et al., 2023;). A learning opportunity for novice mathematics teacher educators refers to any situation that allows them to engage in the process of negotiating meaning, thereby acquiring new ways of thinking that underpin their new professional practice.

In this process, a broker is a more knowledgeable person playing the role of a mediator or bridge between different groups, as in this case, the communities of mathematics teachers, mathematics educators, and researchers in mathematics education. A broker connecting ideas from different communities facilitates the flow and exchange of information, making explicit the implicit knowledge of one group to another. This role enables participants to access diverse knowledge that might otherwise be unavailable. Furthermore, the broker often mediates between different perspectives when moments of tension or misunderstanding arise.

From a sociocultural perspective on learning, the process of negotiating meaning is how novice mathematics teacher educators interact to develop a shared understanding of their new practice. That is to say, as a way to construct meaning together with others of the role played by the different registers of practice in the PLTs (the mathematics secondary problem and the students' answers), and about how the participating teachers' learning is understood. The process of meaning negotiation enables novice mathematics teacher educators to organize their practice around key ideas and relevant concepts, illustrating the reification process (Llinares, 2002). The reification process refers to the process by which abstract ideas, experiences, and social relations are transformed into artifacts by the shift of making explicit what is often implicit in the experience. From this perspective, the reification process can be the focus of the study on how the background profiles of novice mathematics teacher educators determine the targets of their learning. Wenger (1998) refers to the reification process as:

the process of giving form to our experience by producing objects that congeal this experience into 'thingness'. In doing so, we create points of focus around which the negotiation of meaning becomes organized ... A certain understanding is given form. This form then becomes a focus for the negotiation of meaning. (pp. 58-59).

[...] Any community of practice produces abstractions, tools, symbols, stories, terms and concepts that reify something of that practice in a congealed form ... with the term reification I mean to cover a wide range of processes that include making, designing, representing, naming, encoding, and describing, as well as perceiving, interpreting, using, reusing, decoding and recasting. (p. 59).

When novice mathematics teacher educators work toward shared goals such as the design and enactment of a professional learning task into a practice-based professional development program, they endow meaning to the mathematical secondary school problem, to the students' answers, like practice registers from their different background profiles, and also endow meaning to participating teachers' learning. From this stance, the reification process creates a focus around which to negotiate the meaning (Llinares, 2002). In this process, when the initial goals are questioned during the design of the PLTs or while reflecting on their enactment, there is a possibility that the PLT's goals can become reified. Negotiating the focus of inquiry in MTE practice involves refining the issue of MTE practice and determining what instances of these practices can be leveraged to illuminate what is involved (Van Zoest & Levin, 2021).

From Wenger's perspective (1998), this learning involves three interconnected processes: i) an increasing engagement with evolving forms of mutual engagement, such as the ability to engage with other members and respond in kind to their interactions, and discover how to engage in practice; ii) understanding and tuning an undertaking of the community, struggling to define the undertaking and reconciling conflicting interpretations of what the undertaking is about, and iii) developing the repertoire, styles and discourse of the community, renegotiating the meaning of different elements, producing or adopting tools, artifacts, and representations.

From this perspective, the collaboration opportunities among different novice MTEs, provided by their participation in the design and implementation of professional development, may be critical in generating a process of meaning negotiation, allowing the identification of critical points for the MTE's learning.

With these conceptual references, we posed the following research question: *What characteristics of novice mathematics teacher educators' learning opportunities are possible to identify when they design and implement a practice-based professional development for practicing mathematics secondary teachers?*

METHOD

We have carried out a qualitative-interpretative study to understand the subjective experiences and perspectives of the novel mathematics teacher as well as uncover the meanings that participants attribute to their lives and experiences.

Participants and Context

Data is collected in a practice-based development course as a real-world environment where the novel mathematics teacher educators participate as instructors. The practice-based professional development (PBPB) was delivered in two cycles in 2023 and 2024. Each cycle consisted of designing, implementing, and analyzing. The PBPB was conducted by eight MTEs with different academic and professional profiles, and it aimed to provide professional learning opportunities for in-service secondary mathematics teachers in terms of rethinking their professional practices (teaching practices) to implement new actions. The emphasis in this course is on the need for participating teachers to design and experiment with innovative teaching approaches for helping students develop new ways of mathematical thinking while teachers reflect upon their teaching practice. The design-oriented perspective adopted defines three levels of involvement for mathematics teacher educators (Chen et al., 2018). Our data comes from the first cycle (2023) (Figure 1).

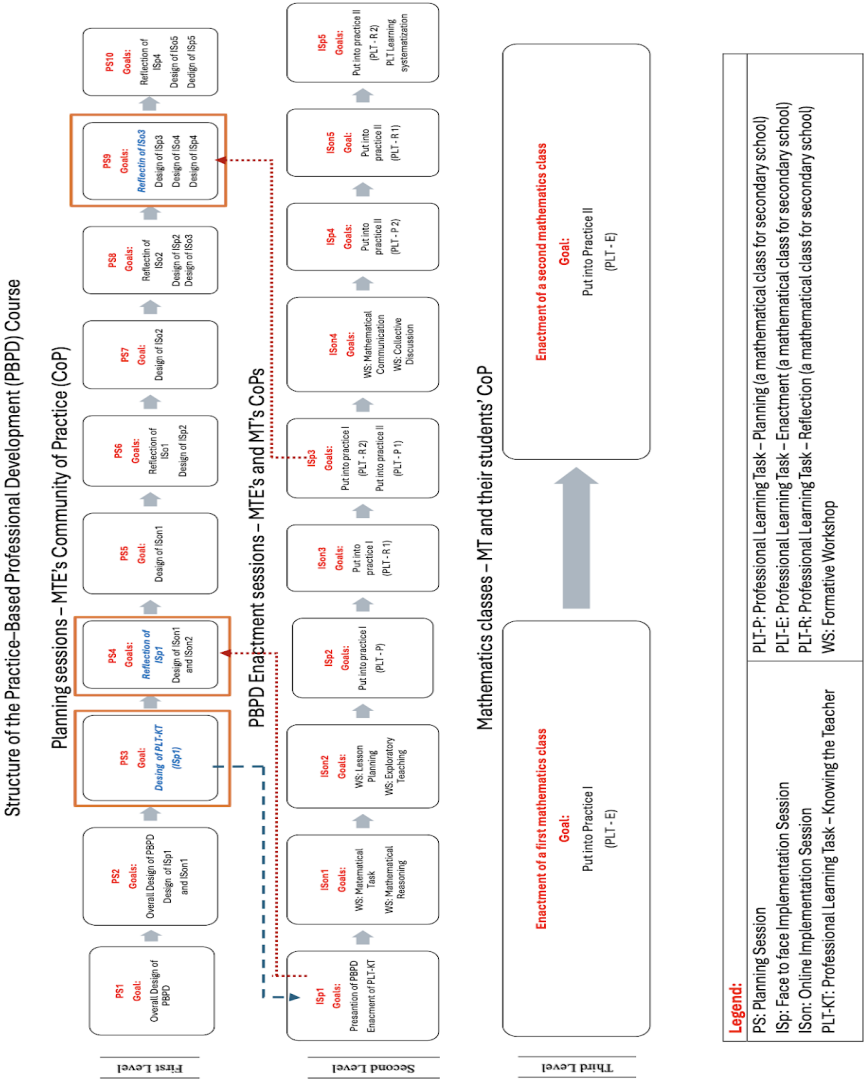
In the first level of the PBPB, ten online planning and analysis sessions (two hours long each) were conducted and recorded. Design and analysis sessions were interspersed with implementation sessions (second and third levels). For example, planning sessions 2 and 3 (PS2 and PS3) were designed to create the first professional learning task, while the face-to-face implementation session 1 (ISp1) was the actual implementation session, and planning session 4 (PS4) aimed to analyze how the implementation worked.

In the sessions aimed at designing PLTs, we had two focuses. First, the focus was on a mathematical secondary school problem (Appendix A) that initially mirrored the different specific mathematical practices of secondary school students linked to relational thinking and could support an exploratory teaching perspective in secondary education. Second, we selected students' answers that highlighted the mathematical structure of the problem and the different features of the students' mathematical thinking, considering the relationships between mathematical lesson goals, students' mathematical thinking, and the nuances of exploratory mathematics teaching. The PLTs' structure and content vary according to the purpose of the session. Two sessions aimed to identify and map participants' knowledge, while four sessions had the goal of introducing and exploring new ideas (e.g., the cognitive demand of problems, introducing exploratory mathematics teaching

principles, and characterizing students’ mathematical reasoning). Finally, two sessions aimed to promote the “put into practice” of the ideas (plan a lesson).

Figure 1

Structure of the practice-based professional development.



The planning sessions were scheduled for two-hour online work meetings. During the online meeting, the group of MTEs shared and discussed ideas, and they could continue working after the online meeting in subgroups or individually, depending on who would be the facilitator to lead the next implementation session, and what topic would be addressed.

The participants in our study are three novice MTEs, chosen from among eight MTEs forming the group, following two criteria: diverse academic and professional backgrounds. The three novice MTEs (Mila, Rose, and June, pseudonyms) have different academic backgrounds and professional trajectories. Mila and Rose are mathematicians with master's degrees in pure mathematics, while June is a mathematics teacher with a master's degree in mathematics education. They all hold a PhD in mathematics education. Mila has professional experience at all levels of education, including graduate school. Rose has extensive experience only in higher education, while June began her career as a higher education teacher less than five years ago. Regarding research, Mila has experience developing research projects and supervising master's students, while June and Rose have not yet developed these skills.

One expert mathematics teacher educator (the first author of this paper). The expert's motivation for participating in the research work was to gain an understanding of the characteristics of novice MTEs' learning when they work together in designing and enacting professional development for secondary mathematics teachers. These group of MTEs can be considered as a community of practice in the sense of Wenger (1998), since they share a common concern, such as the education of mathematics secondary teachers, taking advantage of opportunities provided by the shared design and implementation of a professional development (as a joint enterprise) by mutual engage in negotiating shared understanding and produce a shared repertoire of communal resources (e.g., language, artifacts and stories, and finally a set of professional learning tasks). Furthermore, this design emphasizes the novice MTEs' multiple roles as designers of the workshops, facilitators, and learners by reflecting on their practice (Chen et al., 2018).

Three novel mathematics teacher educators and an expert educator is a small but purposeful sample. Our goal was to gain a deep and rich understanding of learning opportunities for novel mathematics teacher educators. We intentionally select participants who have specific

characteristics, experiences, and different profiles related to the new practice and we assumed that they provided relevant data for our study.

Analysis

This study is a qualitative-interpretive research (Creswell, 2013) and data come from the transcriptions of ten video-recorded planning sessions (sessions aimed at designing and reflecting on the enactment). We focused on what four MTEs with different profiles and expertise said and did in relation to the design, implementation, and post-reflection of the professional learning tasks. We explore choices of teaching materials and the resources they opt for and examine issues such as changing perspectives and negotiating relationships. We identified features throughout their work, which were determined through iterative collaboration. For this reason, the focus of our research is on understanding mathematics educators' learning in practice, not generalizations.

The opportunity to review focal instances using the video record of design and reflection sessions allowed characterizing learning opportunities for novice MTEs, making learning visible through collegial inquiry enhanced by the design of PLT. Our analysis unit consisted of moments of interaction that reflected issues puzzling to the group of MTEs constructing the focal instance of MTE practice in context, where MTEs reflected on artifacts of practice through collegial conversations. We referred to these moments as critical events (Hallen-Halloun & Ayalon, 2025), defining them as opportunities for interaction among participants with different background profiles, and for the negotiation of meanings to be shared. The operative criteria used for the identification of the critical events were: (i) the presence of the four MTEs who are the focus of analysis in this article and (ii) having interactions about the planning, as well as reflections on situations that occurred in implemented session and that could be rethought for later ones and illustrating different initial approaches. The analytical process followed the following steps.

First, the expert MTE (first author) revised the ten video-recorded planning sessions (Figure 1) by watching them, listening to recordings, reading transcriptions, and organizing analytic notes (Roth, 2005) to identify episodes that could be considered critical events for the MTEs' learning. The critical events illustrate how the different background profiles of novice MTEs seem to influence their participation, and the meanings generated.

These events were discussed with another researcher (the second author of this paper) to determine to what extent the event illustrated features that could help us understand how novice MTEs interpreted the professional learning task as a learning tool and how they reflected on their enactment. This process allows us to identify themes that emerge from data, leading us to think about characteristics of learning opportunities that could be inferred, taking into account the increasing engagement, how they are understood, and tuning the enterprise and how they contribute to the shared repertoire, so, generating some features that could illustrate the MTEs' learning. In this process, three foci of negotiation of meaning were identified:

- About the mathematical practices such as generating particular cases, organizing information, conjecturing a pattern, and generalization... through explorations of mathematical problems. This focus is mediated by the mathematical relationships in the mathematical secondary school problems used. This focus also meant to consider the difficulty of mathematical problems for participating teachers in order to facilitate their participation.
- About the structure of the professional learning tasks (PLT): goals, content, prompts, and the practice registers used, such as characteristics of the student's answers.
- About management of the implementation of the PLT: scheduled, organization of mathematics secondary teachers' work, and the mathematics teacher educator's role.

Second, we systematically examined these focuses in various critical events to support or reject our initial assumptions. This second analytical step allowed for the identification of features of learning opportunities for MTEs, refining the writing of some characteristics, integrating others, and rejecting some (Roth, 2005). The characteristics identified illustrate how meaning is generated for novice MTEs during the interaction process through reification processes. For example, the meaning of working with secondary mathematics teachers as learners, and that the MTEs need to engage them with new ideas, such as exploratory teaching (e.g., about the role that the mathematical problem should play in the PLT, and about the meaning of exploratory teaching in secondary education). Furthermore, on how to enact different teaching practices to manage the mathematical discourse in the collective discussions (e.g., about the different roles the facilitator should play in the enactment of the PLT), and about how to help mathematics secondary teachers rethink their practice.

Throughout the process of analysis and refinement of characteristics, different critical events can illustrate several aspects of the same characteristics, thus allowing us to perform internal validation of the results generated (themes). The themes are recurring insights that emerge from the data, allowing us to go beyond merely describing what participants said and to imply interpretation and analysis, thereby uncovering the underlying meanings and perspectives (Roth, 2005). These themes represent a coherent pattern across the dataset, providing a deeper understanding of MTEs' learning. Three of these features are described in the result section: (i) Problematizing what has been assumed; (ii) Seeking a balance between cognitive, social, and affective aspects; and (iii) the role of the expert MTE as a broker.

Ethics Statement

This study has received consent from the research subjects, including both lecturers and teachers, who signed an informed consent form (ICF). This research was approved by the research ethics committee of the institution where it was carried out, whose process number is 73768123.8.0000.5594. Through this statement, we release *Acta Scientiae* from any resulting consequences, including full assistance and potential compensation for any harm experienced by research participants, in accordance with Resolution No. 510, dated April 7, 2016, from the Brazilian National Health Council.

RESULTS

We describe three characteristics of MTEs' learning opportunities: (i) Problematizing what has been assumed; (ii) Seeking a balance between cognitive, social, and affective aspects; and (iii) Modelling the expert MTE's role as a broker.

The first feature, problematizing what has been assumed, focuses on how the MTEs problematize previous practices (as mathematics teachers and as mathematics teacher educators), and it is made explicit when they discuss the design of a PLT and how to launch it. This involves considering how the mathematical secondary school problem can support secondary mathematics teachers in rethinking their practices and determining when to introduce the secondary school students' answers for discussion, in order to generate opportunities to identify students' specific mathematical practices and difficulties.

The second feature is focusing on how the MTEs strive to strike a balance among cognitive, social, and affective perspectives regarding the goals of the PLT. This characteristic is defined by how MTEs select the mathematical problem in the PLT, how to launch the PLT and organize the implementation sessions, and how they consider the emotions and feelings of participating teachers. This feature highlights how MTEs consider the principles of fairness when treating participants in professional development.

The third feature focuses on two moves of the expert MTE as a broker. First, facilitating knowledge flow connecting novice MTEs with research results in Mathematics Education, seeking to utilize research results as a tool, and trying to connect the novice MTE with the mathematics education research community. Second, mediating different perspectives, values, and norms that novice MTEs make explicit during the negotiation process and that might generate tension, misunderstanding, or conflicting views between participants.

We present the three characteristics and discuss as they define learning opportunities using evidence from three critical events focused on: (i) rethink the structure and launch of the PLTs (planning session 3); (ii) nature of mathematics secondary school problems that should be used in PLT (planning session 4), and (iii) the role of mathematics education research results in the design and implementation of practice-based professional development intervention (planning session 9). We describe and interpret the three critical events to illustrate how the interplay among the three characteristics occurs, thereby defining learning opportunities for novice MTEs. These critical events illustrate how the expert MTE and novice MTEs collaborate to share expertise, plan the professional development, analyze data, and solve everyday challenges.

Critical Event 1: Rethinking the structure and implementation of PLT. Problematising what has been assumed

This critical event focuses on rethinking the structure and implementation of PLT, problematising the assumed practice. The dialogue begins in Planning Session 3 (Figure 1), with Mila presenting how the PLT (Appendix 1) should be launched in the first implementation session, taking into account how the other PLTs had been launched in previous formative processes. At this time, the PLT had two mathematical problems about the identification of patterns (Garden A: $p1 = 10n + 2$ and Garden B: $p2 = n^2$) and

the initial goal was to elicit teachers' mathematical knowledge of recognizing patterns and generalizations as examples of algebraic thinking (addition, questions was inquiring about features of students' mathematical thinking). However, in the previous professional development, Mila had used two PLTs (in two sessions of four hours each), although the prompts regarding students' answers or teachers' future actions were the same. Mila pointed out this fact and shared their thought about how the new PLT with two problems should be launched (solve the problem in small groups, and next discuss in the whole group, before discussing the students' answers).

Mila: It's like this, look. We need to set up the PLT. We need to set up a PLT for the "Rosacea Area" problem and a PLT for "Embellishing the Garden" problem. Both need to have the same questions, the same approach, which is after the analysis, okay? Actually, what I brought here, that we have combined PLT 1 and PLT 2 [referring to combining two PLTs into a single new one]. (...) What was I thinking? First, we give them the mathematical task. But now there is just one detail. [We cannot launch the two problems to be individually solved and then...], they [participating teachers] get together in a group, and then there's a plenary session. There's no way to do that. And then go and look at the students' records. So, I was thinking of doing the math problem collectively.

Mila considered that discussing the resolution of the two mathematical problems, examining students' mathematical practices, and then exploring how to develop exploratory mathematics teaching can be challenging for teachers and is time-consuming. In this situation, and with the intention of challenging MTEs to think about new possibilities for launching the new PLT, Alessandro asks them to consider the goals of the PLT, which include analyzing mathematical topics in the problem and discussing how to enact them in the secondary classroom.

Alessandro: First, before we move on, you [Mila] said that there is not enough time for them [the teachers] to solve it individually and then discuss it collectively. (...) But I think we can adapt this [proposing that the implementation of new PLT would not need to be done in the same way as in previous training]. First, I think that we should think about the mathematical problem. What do we want them [the teachers]

to do? ... [We want] to explore a little the mathematical knowledge that he/she [the teacher] will mobilize to solve the task. (...) And then we would have an additional hour to work on the other dimension [referring to exploring different ways in which this type of problem could be used in secondary mathematics teaching by the participating teachers].

This intervention presents an opportunity to align the PLT's goals with the time constraints to manage the challenge of launching the new PLT. Consequently, it opens the possibility of critically examining and revising established practices by considering the new specific context and the goals of PLT, "what do we want them [the teachers] to do?". With this action, expert MTE brokers between the experiences of novice teachers and the realities of the profession, focusing on the goals intended each time. So, the expert gives a concrete form to the target in the discussion, making the goals of LPT become objects to interact with, discuss, and refine. The interaction that followed illustrates how MTEs managed this challenge, considering various viewpoints as an example of the reification process of PLTs' intended goals from the problematization of assumed practices.

Rose: Won't there be too little time to solve both mathematical problem?

Alessandro: Not both, just one.

Rose: Oh, just one.

Mila: Yes, actually... I'm against this idea [of using only one mathematical problem], but that's okay. We, in the other courses [referring to other formative processes] (...) we divided it up [referring to participating teachers]. Those who were in elementary school only did the elementary school's task. Those who were in high school only did the high school's task. Do you understand? So, no one would do both; each one would only do one.

June: Ah, will the teachers be divided into groups according to the grades they teach?

Alessandro: I would suggest that, at this moment, we don't divide them (...) [since this time the participants were all secondary teachers]

Mila: I agree too, I don't like this thing of having a plenary session when you haven't seen the problem [referring to using two mathematical problems]. I don't think anyone pays attention.

Rose: (...) Because I think that if there are two mathematical problems, when we hold a plenary session to discuss ideas, since time is short, we will have to have two moments to discuss different ideas. And we will waste time with that, instead of exploring the ideas in a single problem.

June: Yeah, I think it's cooler if everyone discusses the same thing.

Mila: Then I think one thing, we can look at both mathematical problems and decide [referring to the two mathematical problems previously selected in PS2].

In this interaction, Rose is unsure if using two different mathematical problems, as Mila previously suggested, would leave teachers with insufficient time for a deeper exploration of the problems. Alessandro, as an expert, emphasizes the intended goal for teachers to be able to think about how to use this type of problem in secondary teaching to support students' specific mathematical practices. In this interaction, the group of MTEs defined as a joint enterprise (the target): thinking about the launch, considering the intended goals of the PLT, having time to discuss the demand of the mathematical problem, and giving teachers the opportunity to rethink their practices.

Next, during Planning Session 4 and reflecting on the implementation, Rose reports a conflicting situation between her and Mila in the first face-to-face implementation session. This conflicting situation arose from the differing perspectives on Rose's actions during this session. The explicit differences in how to act in the management of the PLT trigger a series of discussions and negotiations of meanings in the group, leading to rethinking the assumed practices:

Rose: One thing that made me a bit upset on Friday [referring to face-to-face implementation session 1], was that when I was going around the groups. I wanted to explain this to Márcia, right, ... I wasn't interfering in the answers [of the participants]. I was trying to listen to what they were saying.

(...) I wasn't telling what they had to do, and I wasn't discussing ideas, right?

So, it is not clear for me what we as educators should do. No one told me what to do. If I had been told what to do at the beginning [Rose is questioned about the lack of previous discussion about the role of MTEs], I wouldn't have said it, right? (...) Maybe we only need to have better guidance on how to act, right? Because otherwise it gets annoying.

Mila: No, Rose, sorry [referring to the incidents related by Rose]. (...) we tell you the intended goals of each PLT [She is talking about the intended goals and as a consequence how PLT should be launched and enacted]. [In] each PLT, our intervention is different [since] we have to take into account the prior knowledge [of teacher participating] (...) [The goal of this first part of the PLT] is to get to know them. That is, it is more for us to listen than for us to interact with them (...)

Mila tries to explain to Rose – and the other MTEs – the different roles and actions that MTEs should perform depending on the context and the PLT's intended goals. Mila tries to connect the reflections on the role of the MTE in a specific context to the PLT goals (the target of the interaction). These interactions illustrate how MTEs make explicit the target of discussion as a means of negotiating meanings to challenge previously assumed practices. For some PLTs, for example, where the goal is to identify and map teachers' prior knowledge, educators should assume a more observational role, with few interventions. On the other hand, when the PLT aims to provide opportunities for teachers to build new knowledge or implement new practices, the teacher educator's role is to mediate interactions between teachers and between them and the PLT. This interaction between Mila and Rose illustrates how common language can be seen as a tool for expressing needs, clarifying understanding, and persuading others. It also shows how the MTEs work to understand each other's perspectives, clarify ambiguities, and build a shared understanding of new practices. Sharing meanings about what happened in the face-to-face implementation session allowed them to consider prospective practices, including what they likely would like to happen in the future, especially in the subsequent implementation sessions that would take place over the four months of the formative process.

These interactions illustrate how MTEs seem to move away from established ways of thinking and interacting, as they play a new role as MTEs.

These interactions demonstrate a process in which MTEs actively choose to adopt new practices influenced by social interactions and the way they utilize the PLT as a cultural tool. The interaction describes how the situation exposes the novice MTE to new perspectives and ways of doing things, prompting them to reconsider their existing practices. We can understand these critical events as examples of how the design and use of PLTs in professional development mediate the thinking and actions of novice MTEs. That is to say, resigning from old practices involves adopting and internalizing a new role as an educator.

Critical Event 2: The nature of mathematics secondary school problems that should be used in PLT. Considering the balance between cognitive, social, and affective aspects.

One characteristic of MTEs' learning opportunities surfacing different critical events was to consider the balance between cognitive, social, and affective aspects. This feature emerges when MTEs discuss the nature of mathematical problems in secondary school in the PLT. For example, in Planning Session 3, Rose questioned using one or two mathematical problems due to the time and dynamics of carrying out during the first face-to-face implementation session (as has been discussed in Critical Event 1). This issue generated a negotiation of meanings focused on the tension between the cognitive and social dimensions of the MTE's practices (choosing a mathematical task according to the teacher's educational level –cognitive dimension– and having enough time to discuss the potential of the mathematical problem –social dimension–).

In the excerpt discussed above, we also observe a negotiation of meanings to balance the social and cognitive dimensions. Rose's initial questioning leads Mila to externalize her initial discontent with the proposal to use a single mathematical task (Yes, actually... I'm against this idea). Mila supported her argument from her previous experience as a facilitator of secondary teachers (We, in the other courses, we divided it up. Those who were in elementary school only did the elementary school's problem. Those who were in high school only did the high school's problem). However, Mila's concern is related to the cognitive aspects of choosing the mathematical problem, namely, that teachers teaching at an educational level should solve mathematical problems related to this level. In contrast, Rose's initial concern was on whether there would be enough time for teachers to discuss the potential of the mathematical problem. The interaction between

June and Rose (lines 7-9) can be understood as a search for a shared meaning. This alignment considers a balance between the cognitive (e.g., that the mathematical problem is more suitable for the teaching level at which the teacher works) and the social (e.g., that there is enough time for teachers to do a good job with PLT during implementation sessions 1).

After the group decided that the PLT would contain only one mathematical problem, they began negotiating which problem to choose. This choice again interlaces the cognitive and social dimensions, trying to align with the goal intended:

Alessandro: (...) Let's first think like this: "If we had to choose one of the two problems for teachers to work on, so we can try to raise some mathematical knowledge from them and then from the students, which of the two would be the most interesting?"

Rose: I think what we need to think about is whether we want them to have difficulties or not. Because if we want them not to have difficulties in mathematics, I think that would be "the garden task". Now, if we want them to think more about the subject [Mathematics], for them to reflect, I think the most complex one is the "rosacea area task".

Alessandro: Which one would you choose?

Rose: I think that, at first, since they are just starting out with this idea of an investigative proposal [referring to the exploratory teaching approach], I would go with "the garden task". I don't know, that's my [opinion].

Alessandro: Great. Mila and June, what do you think about what Rose said?

Mila: So, I'm going that way too (...) I agree with Rose, because as we want, first motivate them to take the course [the formative process]. After they think about the knowledge they have, after they think about the didactical and mathematical knowledge, after they think about the student, in all this PLT that we are doing, I think that "the garden task" is more appropriate.

Alessandro: Great. June, what do you think?

June: Thinking about teachers, I believe that, to motivate them, “the garden task” I think makes it easier for them to really respond.

Alessandro: I think so too. I think we shouldn’t scare them on the first day, maybe by giving them a math problem that they have a lot of difficulty solving. They’ll feel embarrassed, they’ll feel intimidated.

In this critical event, we observed that the negotiations between novice MTEs are divided between cognitive and social aspects when choosing the appropriate mathematical problem. We can even observe that there is a predominance of the social aspect, as they demonstrate concerns about the cognitive demands of the mathematical problem, serving as a gateway to the formative process. For example, we observed Rose saying that she would choose the mathematical problem “Beautifying the Garden” because the teachers “are just starting out with this idea of an investigative proposal [referring to the exploratory teaching approach], that is, their concern with affective aspects of not discouraging teachers right at the beginning of their formative process. The same occurs with Mila, which indicates “I agree with Rose, because as we want, first motivate them to take the course [the formative process]”. Finally, June also agrees with her colleagues, pointing out, thinking about teachers, I believe that, to motivate them, ‘the garden task’, I think it makes it easier for them to really respond.”

This critical event illustrates how the MTE incorporates aspects of social context, affective perspectives, and cognitive issues when they interact to determine the type of mathematical problem they should choose. This illustrates a dynamic, interactive process through which MTEs in a social setting arrive at a shared understanding about the suitability of a problem as a collaborative construction of meaning.

Critical Event 3: The role of mathematics education research results in the design and implementation of practice-based professional development intervention. The expert acts as a broker, mediating interactions

One feature defining the learning opportunities of novice MTEs was the role of the expert MTE as a broker, mediating interactions between novice MTEs and creating opportunities to internalize knowledge from mathematics education research as cultural tools, thereby mediating the resolution of

tensions (as described in critical event 1). Here, we illustrate the first aspect of the role of an expert as a broker, specifically how to utilize research results from mathematics education, considering two key aspects. Firstly, from Planning Sessions 3, he focused on the reasons behind the prompts in the PLT (aligning the structure of prompts to cognitive skills of professional noticing, attending to, interpreting, and decision-making). Secondly, during Planning Session 9, he introduced the idea of utilizing the knowledge accumulated by educators.

In Session 3, MTEs discuss the structure of the PLT and strategies for addressing the goal of eliciting teacher knowledge. In this situation, Mila focuses attention on the type of questions in the PLT (about the cognitive demand of the problem and about the students' strategies and difficulties), and Alessandro tells MTEs what justifies these prompts and underlines their link to the cognitive skills of professional noticing.

Mila: (...) I've included the name of PLT2 here so that I can get it right later. So, it's like this. So, there's PLT, which is about teachers' knowledge and about students' answers and strategies (...)

Alessandro: For them [the teachers] to answer what strategies they think the students could use; what mistakes and difficulties the students could make; and what actions they would take to support these students to overcome these difficulties or rethink or overcome these mistakes. Why am I using these three questions? So that we can later, when analyzing, use the concept of "Noticing", that is, identify, interpret, and seek a solution.

(...)

So, these three ideas would be left. The first question is for them to think about what strategies the students could use. And then we will see if they will use the same ones as theirs or if they will bring another one. Then, what mistakes and difficulties could the students present? And what actions would you take to help? With these three questions, we are covering the identify skill. What does identify mean? Pointing out. What strategies and what mistakes or difficulties the student may have. Interpreting, to be able to think in actions,

he [the teacher] has to interpret that difficulty. ... What actions... and justify.

Alessandro focuses MTEs' attention on the reasons for PLT prompts and how these prompts should be managed during the implementation. With this intervention, Alessandro connected the design, launch, and implementation of PLT to research results in teacher education, specifically to the concept of noticing and introducing new elements (shared repertoire) from research and facilitating the flow and exchange of thoughts. With this action, the expert helps to make explicit the implicit knowledge that underpins the design of a PLT.

In the same way, in Session 9, in which the facilitator should launch the task for participant teachers of designing a mathematics lesson following the principles of exploratory mathematics teaching, Alessandro proposes to use a mathematical problem, and a lesson plan used in previous formative processes. However, Mila questions this suggestion and Alessandro supports his argument by bringing to consideration ideas from the mathematics education research community. This interaction put the target of interaction among MTEs on the question of how teachers (participants in formative processes) should use the knowledge accumulated by the community of educators, and how teacher educators could consider this information.

Alessandro: And in the afternoon [of the implementation session 3], we can have two moments. One moment that could be more concise, which would be to present some lesson plan, some problem [referring to materials used in previous formative processes] (...)

Mila: I have a question to ask you. What is the purpose of taking a lesson plan as if it were a model? And is that what we want to pass on to teachers?

Alessandro: So, look. (...) Mila, although you are right about the issue of "proposing model", when you read the text by Jinfa Cai that we are going to discuss at ForMatE [research group the MTE participate in] next week [researcher in the area of mathematics education], one of the things he emphasizes, and not only him, but other authors as well, is that in education, we usually start from scratch. We never take advantage of what we already have to build on. So, he

emphasizes the importance, often, of us continuing to use certain types of problems and refining each one better each time. Why? Because it becomes a contribution, a legacy that we leave to other generations.

When answering Mila's question, "What is the purpose of us taking a lesson plan as if it were a model?", Alessandro seeks to negotiate the meaning of "using the same lesson plan not as a model", but in the sense of achieving a shared repertoire arising from research results in the field of mathematics education. Introducing knowledge from mathematics education as elements of discussion among MTEs is a way in which the expert MTE (the broker) generates possibilities to share conceptual resources. In these examples, the conceptual reasons have been the reasons behind the prompts in the PLT and behind a proposal of action. These extracts illustrate the role of an expert MTE in facilitating connections and knowledge transfer between the MTE practitioners and mathematics education as a social science. By connecting individuals and groups, the expert MTE, as a broker, helps build social capital within a network. He enables novice MTEs' access to diverse knowledge, resources, and opportunities that might otherwise be unavailable.

DISCUSSION AND CONCLUSIONS

This paper aims to identify characteristics of learning opportunities for novice mathematics teacher educators with diverse backgrounds participating in a practice-based professional development program for in-service secondary mathematics teachers. Novice mathematics teacher educators are involved in planning, facilitating, and reflecting on professional development for secondary mathematics teachers. We adopt a sociocultural perspective on mathematics teacher educators' learning (Goos, 2020) as a move from being a novice MTE to being an experienced MTE through interactions with others, developing forms of collaborative analysis and interpretation. We consider this approach as an instance of practice-based teacher educator learning, as the MTEs learned how to cope with the demands of becoming MTEs in transitioning from their background profiles to expertise as MTEs in a professional development program for secondary mathematics teachers. We have identified three features of learning opportunities for novel MTEs: (i) Problematizing what has been assumed; (ii) Seeking a balance between cognitive, social, and affective aspects; and (iii) The role of expert MTE as a broker. These features have been identified allowing novel mathematics teacher educators to reflect on the actions of others, receive feedback on their facilitating skills, and observe alternative

approaches by participating in a professional development program (Chen et al., 2018; Olanoff et al., 2021). We argue that the emergence of these features can be attributed to the novice mathematics teacher educators' profiles and the specific nuances of the setting (goals and values of the program). In this sense, we argue that our findings contribute to advance in our knowledge in the field of mathematics teacher educators' learning (Beswick & Goss, 2018; Goos, 2020; Jaworsky, 2008) in two aspects: firstly, providing empirical support to features of learning opportunities of novel MTEs and, secondly, showing the strength of sociocultural perspective of learning to explicate the learning of novel MTEs.

The three features can help us determine how teacher educators acquire expertise and how the setting matters in their learning (Knight et al., 2014; Wu et al., 2020). Hence, we argue that the setting and reflection on practice can be considered factors influencing the emergence of learning opportunities.

First, the practice-based teacher education program created a dynamic learning environment where novice mathematics teacher educators (MTEs) could cycle through designing, enacting, and reflecting on professional learning tasks (PLTs). This iterative process is crucial in understanding the MTEs' learning and the development of their practices, highlighting their ongoing self-reflection (Chen et al., 2018). Second, continuous reflection – both on their own practice and on the design of professional learning tasks (PLTs) – highlights the value of collegial reflection on different approaches to mathematics teaching (van Zoest & Levin, 2021). Hence, the design of PLTs addressed to reflect on new ways of teaching appears as a key factor in the emergence of learning opportunities for the novice mathematics teacher educators. This fact highlights the role of tool design as a mediating factor in the learning of novice MTEs (Chen et al., 2018; Superfine & Pitvoreck, 2021). The relevance of setting and the reflection, as factors influencing the emergence of learning opportunities for novice MTEs, was defined considering as key points of the discussion in the critical events, the relationships between goals intended, the type of problem and students' answers selected. This fact shifts our attention to the relevance of joint reflection as negotiating of meanings, illustrating the reification of ideas such as the relationship between the launch of the PLTs and its influence on how participant teachers can develop new ways of thinking about the teaching of mathematics.

The dynamic interplay of these features defines learning opportunities for the novice MTEs conveying the idea of learning defined by different, changing aspects over time, constantly being shaped by its diverse components that contribute in varying ways at different times. We have illustrated that these features do not always contribute equally or simultaneously to novice MTEs' learning by describing and interpreting three critical events determining the relationships between research (how do MTEs learn?) and practice (what do MTEs do?) (Knight et al., 2024; Forzani, 2014).

The first feature, *Problematising what has been assumed*, illustrates the process of adapting previous practice to a new setting. This feature shifts novice MTEs' attention to different aspects of practice. The second feature, *Seeking a balance between cognitive, social, and affective aspects*, focuses on how support participating secondary mathematics teachers as learners emphasizing how MTEs learn to think about scaffold practicing teachers. Furthermore, this feature highlights the importance of paying attention to participants' learning as a factor influencing MTEs' learning (Schwartz et al., 2021; Karsenty et al., 2023). This feature underscores the importance of selecting adequate mathematical problems and students' answers to share, as they highlight the exploratory mathematics teaching approach intended. The third feature, *Modelling the expert MTE's role as a broker*, illustrates the extent to which teacher educators need to be good consumers of research (Loughran, 2014). This feature emerges when the focus of the interaction is on how to generate arguments to justify decisions, thereby situating MTEs' learning within the development of pedagogical reasoning. The expert MTE helps create learning opportunities for novice MTEs by translating the target of discussion into different ways of understanding the situation (e.g., emphasizing the role played by results of mathematics education research to understand the new situation). The flow of information in key moments enables the transfer of relevant knowledge, such as the relationship between the structure of the PLT and the cognitive skills of attending, interpreting, and decision-making, derived from professional noticing or the ideas that support the lesson plan aimed at developing exploratory mathematics teaching (Ponte, 2012). We argue that when the expert MTE introduces new perspectives and challenges existing norms, he is helping create learning opportunities for novice MTEs bridging theory and practice.

Our findings highlight the role of iterative collaboration and collegial conversations around of designing PLTs as artifacts of practice in defining learning opportunities for novice MTEs (van Zoest & Levin, 2021) considering the skills of professional noticing (attending to, interpreting, and

decision-making) as a reference to organizing the prompts concerning different practice registers (mathematics problems and students' answers). The collegial conversations around designing artifacts such as PLTs seem to be an effective way to generate learning opportunities for novice MTEs, involving the critical engagement with key questions and issues in practice (Jaworski, 2008; Karsenty et al., 2023; Zaslavsky, 2008). Furthermore, defining the focus of inquiry in critical events involve refining novice MTE practice and determining which instances of their practices can be leveraged to understand that practice (van Zoest & Levin, 2021). In particular, when the focus of inquiry in the interaction among MTEs is how to lead productive discussions and achieve the goals of the professional development (Schwartz et al., 2021).

IMPLICATIONS AND PROSPECTIVE DIRECTIONS

Based on our findings, we can derive some implications for MTEs' learning. First, the identified features of learning opportunities could be explicitly focused when novice MTEs participate in practice-based professional programs to support their pedagogical reasoning (Loughran, 2014). For example, rethink their ways of organizing and understanding learning environments for the professional development of secondary mathematics teachers. Second, we emphasize the importance of diverse background profiles of novel MTEs working together and the role played by a more knowledgeable expert (Wenger, 1998), who serves as a broker between novice MTEs and mathematics education communities. The expert MTE as mediator novice MTEs with the new roles they can play as facilitators, and the research in mathematics education, facilitating the flow and exchange of information making explicit the implicit knowledge that underpins their work as facilitators. Our findings highlight relevant aspects of training proposals that should be considered in professional development programs, in particular, the potential of the structure adopted in the professional program (Figure 1).

On the other hand, our findings generate new research questions such as trying to increase the scope of MTEs involved in different communities of practice, deepen the understanding of experienced teacher educators as brokers between different communities of practice, and seek to understand how some artifacts used (e.g., PLTs) can be understood as boundary objects (Akkerman & Bakker, 2011).

AUTHORS' CONTRIBUTIONS STATEMENTS

Conceptualization, A.R. and S. L.; methodology, A.R. and S. L.; formal analysis, A.R. and S. L.; re-sources, A. R.; data curation, A.R.; writing,

A.R. and S. L. All authors have read and agreed to the published version of the manuscript.

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DATA AVAILABILITY STATEMENT

Anyone making a reasonable request to the first author of the article, AR will be provided with the supporting data for the research findings.

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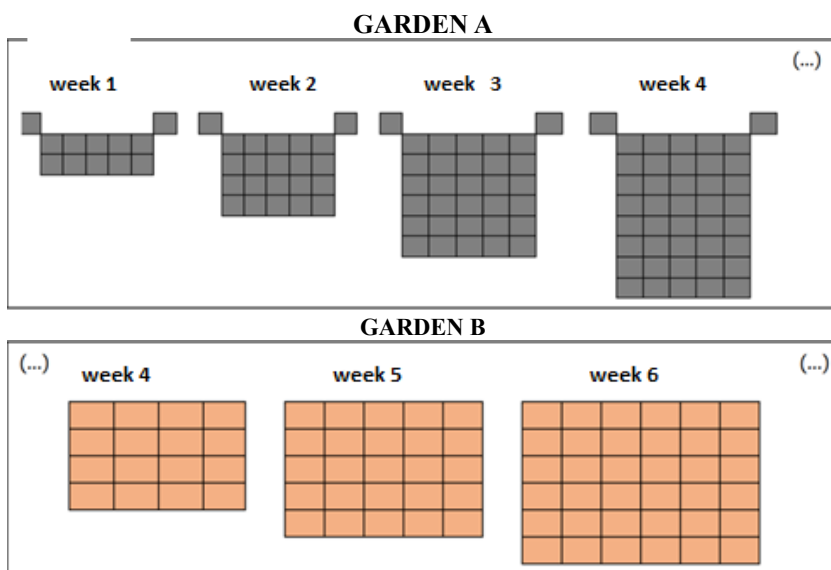
APPENDIX A.

EXCERPTS OF PROFESSIONAL LEARNING TASK (PLT)

1^a Part (Solving Mathematical Problem): Solve the mathematical problems collectively

Mathematical Problem: “Embellishing the Garden”

Joe is a gardener on a farm. On the farm, he has two gardens where he grows plants that he wants to beautify. He wants to plant a plant in each square of the garden, and he want to modify his gardens every week, according to the following sequences:



Considering the planting done each week, answer:

1. Based on **Garden A**, answer:

- 1.1. How many different plants will the garden have in week 5? And in week 20?
- 1.2. Will the garden ever have 512 plants? Explain how you thought.
- 1.3. What could be the general term of the sequence corresponding to garden A?

(A) $n + 10$; (B) $2n + 10$; (C) $10n$; (D) $10n + 2$.

Justify your answer.

2. Based on **Garden B**, answer:

2.1. How many plants will the garden have in each of the first three weeks?

2.2. How many plants will the garden have in week 10?

2.3. Will the garden ever have 512 plants? Explain your reasoning.

2.4. What could be the general term of the sequence corresponding to the number plants in garden B? Explain your reasoning.

3. Considering the **two gardens**:

3.1. In which of them would it be possible to plant 65 plants first? Explain your answer.

3.2. In which of them would it be possible to plant 120 first? Explain your answer.

3.3. After a few weeks, the number of plants in one garden will be three times the number of plants in the other. After how many weeks will this relationship be verified?

Teacher:

If secondary school students were to solve this math task, what strategies would they use?

What difficulties might secondary school students have when solving this task?

How would you help students overcome the difficulties they may present?

Justify your answer.

2ª Part (Analysing secondary school students' strategies and resolutions, and deciding what to do)

Teacher: identify some characteristics that you consider relevant, in the resolutions of each group of students regarding garden A (for example).

Considerando a plantação feita em cada semana, responda:

1. Tendo por base o Jardim A, responda:

1.1. Quantas espécies diferentes de plantas terá o jardim na semana 5? E na semana 20?

• Na quinta semana haverá 52 plantas:
→ Cada semana aumentam 10 espécies de plantas diferentes
 $5 \cdot 10 + 2 = 52$
semana quantidade padrão

• Na semana vinte haverá 202 plantas:
→ $20 \cdot 10 + 2 = 202$

1. Based on **Garden A**, answer:

1.1. How many different plants will the garden have in week 5? And in week 20?

In week 5 there will be 52 plants
Each week increase by 10 plants
 $5 \cdot 10 + 2 = 52$

In week twenty there will 202 plants
 $20 \cdot 10 + 2 = 202$

Considerando a plantação feita em cada semana, responda:

1. Tendo por base o Jardim A, responda:

1.1. Quantas espécies diferentes de plantas terá o jardim na semana 5? E na semana 20?

$a_1 = 0,1 + (n - 1)10$
 semana 5: $a_5 = 0,1 + (5 - 1)10$
 $a_5 = 0,1 + 40 = 40,1$
 $a_5 = 52$
 semana 20: $a_{20} = 0,1 + (20 - 1)10$
 $a_{20} = 0,1 + 190 = 190,1$
 $a_{20} = 202$
 na 5ª semana terá 52 espécies diferentes e na 20ª semana terá 202 espécies diferentes.
 Raciocínio: Pegamos a fórmula da PA e aplicamos as informações que o exercício nos deu.

1. Based on **Garden A**, answer:

1.1. How many different plants will the garden have in week 5? And in week 20?

In week 5 there will be 52 plants and in week increase by 10 plants:
 $a_5 = 12 + (5 - 1) \cdot 10 = 52$; and in week twenty there will 202 plants:
 $a_{20} = 12 + (20 - 1) \cdot 10 = 202$.

Reasoning: We took the A.P. (Arithmetic Progression) formula and applied the information the exercise gave us.

Group B

Considerando a plantação feita em cada semana, responda:

1. Tendo por base o Jardim A, responda:

1.1. Quantas espécies diferentes de plantas terá o jardim na semana 5? E na semana 20?

Na semana 5 terá 52 espécies. E na semana 20 terá 202 espécies.

1. Based on **Garden A**, answer:

1.1. How many different plants will the garden have in week 5? And in week 20?

In week 5 there will be 52 plants.
 And in twenty there will 202.

Group C

Teachers,

How do you interpret the student's solutions from groups A, B and C. Which mathematical knowledge, reasoning and strategies have been used? Justify your answer.

Considering the students' resolutions, what would be the possible actions of the teacher to improve students' reasoning?

Do you identify similarities or differences in students' strategies? If so, what are they?

If the answers are incorrect or incomplete, what would be the possible actions the teacher could take to help students overcome the difficulties they had?

How would you share, with all students the different reasoning present?