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Parametrisation of curves: A case study, from a modelling of complex variable functions

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ABSTRACT

Background: The academic process has become a cycle of repetition and continuous subject failures, a situation that motivates students to abandon their professional education. Mathematical modelling in engineering education has become a tool that allows you to represent, manipulate, and communicate real-world objects through the application of mathematical content. **Objectives**: Simulate processes students find complex; provide elements that encourage hypothesis generation and suggest experiments or validation methods. **Design:** A mathematical model is planned and developed using hypermedia. **Setting and participants:** Students in the fifth and sixth semesters (undergraduate) of three engineering programs—electrical, mechanical, and industrial—who take the advanced mathematics course included in their curricula. **Data collection and analysis:** The implemented strategy sought to address the need to display and manipulate content applications involving complex variable functions, using specific examples of academic qualifications for engineers, whose solutions involve the application of integrals. **Results:** The research showed academic progress in at least three quarters of the selected sample. **Conclusions:** The experience showed the importance of integrating technology into the curriculum. The planned and applied hypermedia were adequate for students to overcome identified difficulties and distinguish the effects of different parametrisations of the same geometric place. Therefore, it is recommended not to provide too much information; instead, to use different formats, such as verbal and analytical formats, videos, graphics, and to organise adequate hyperlinking.

Keywords: Parametrisation of curves; Complex variable functions; Mathematical modelling.

Parametrización de curvas un estudio de caso, desde una modelización sobre funciones de variable compleja

RESUMEN

Antecedentes: Los efectos del proceso académico se han convertido en un factor de repitencia y en una pérdida continua de asignaturas, lo que motiva a los estudiantes a abandonar su formación profesional. La modelación matemática en la formación de ingenieros se ha convertido en una herramienta que permite representar, manipular y comunicar objetos del mundo real mediante la aplicación de contenidos

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matemáticos que lo facilitan. **Objetivos**: Simular procesos que resultan complicados para el estudiante; brindar elementos que conduzcan a generar hipótesis y a sugerir experimentos o métodos de validación. **Diseño:** se planea y desarrolla una modelación matemática mediante hipermedia. **Entorno y participantes:** estudiantes de quinto y sexto semestre (pregrado), de tres ingenierías: eléctrica, mecánica, industrial, que en sus mallas curriculares toman la asignatura de matemáticas avanzadas. **Recopilación y análisis de datos:** La estrategia implementada buscó solventar la necesidad de mostrar y manipular aplicaciones de contenidos que involucran funciones de variable compleja desde ejemplos concretos propios de la formación académica para ingenieros y cuya solución involucra la aplicación de integrales. **Resultados:** La investigación mostró avances académicos en al menos tres cuartas partes de la muestra seleccionada. **Conclusiones:** la experiencia evidenció la importancia de integrar la tecnología en el currículo. El hipermedio planeado y aplicado resultó adecuado para que los alumnos superaran dificultades identificadas y lograran distinguir el efecto que producen las diferentes parametrizaciones de un mismo lugar geométrico. Por ello, se recomienda no brindar demasiada información, pero sí usar diferentes formatos, verbal, analítico, videos, gráficos, y organizar una hipervinculación adecuada para no perderse.

Palabras clave: Parametrización de curvas; Funciones de variable compleja; Modelización matemática.

Parametrização de curvas: um estudo de caso, a partir de uma modelagem sobre funções de variável complexa

RESUMO

Antecedentes: Os efeitos do processo acadêmico tornaram-se um fator de repetência e de perda contínua de disciplinas, situação que leva os alunos a abandonar sua formação profissional. A modelagem matemática na formação de engenheiros tornou-se uma ferramenta que permite representar, manipular e comunicar objetos do mundo real, por meio da aplicação de conteúdos matemáticos que o facilitam. **Objetivos:** Simular processos complexos para o aluno; fornecer elementos que conduzam à geração de hipóteses e sugerir experimentos ou métodos de validação. **Design:** planeja-se e desenvolve-se uma modelagem matemática a partir do uso de hipermídia. **Ambiente e participantes:** alunos do quinto e sexto semestre (graduação) de três cursos de engenharia, elétrica, mecânica e industrial, que têm a disciplina de matemática avançada em seus currículos. **Coleta e análise de dados:** a estratégia implementada buscou atender à necessidade de mostrar e manipular aplicações de conteúdos que envolvem funções de variáveis complexas a partir de exemplos concretos próprios da formação acadêmica de engenheiros e cuja solução envolve a aplicação de integrais. **Resultados:** a pesquisa evidenciou avanços acadêmicos em pelo menos três quartos da amostra selecionada. **Conclusões:** a experiência mostrou a importância de integrar a tecnologia ao currículo. O hipermeio planejado e aplicado revelou-se adequado para que os alunos superassem as dificuldades identificadas e distinguissem o efeito produzido pelas diferentes parametrizações de um mesmo lugar geométrico. Por isso, recomenda-se não fornecer demasiada informação, mas sim

utilizar diferentes formatos, verbais, analíticos, vídeos, gráficos, e organizar uma hiperligação adequada para não se perder.

Palavras-chave: Parametrização de curvas, Funções de variável complexa, Modelagem matemática

INTRODUCTION

The use of mathematics in daily life fosters skills and abilities that enhance people's performance, including logical reasoning, competence development, and creativity. Those are elements that enable engineering students to introduce mathematical strategies during their professional training, where they can experience mathematics applicable to their context and observe it as a useful and effective discipline. Therefore, mathematical modelling bridges students' everyday experiences and mathematics. Despite this, in the Colombian educational environment in the last decade, higher education teachers have been perceiving an almost automatic promotion of students from various careers, in subjects typical of advanced mathematical thinking (PMT), which when taking subjects such as advanced mathematics, in which numerical analysis is used as a strategy to face certain engineering situations or mathematical problems that require numerical methods to obtain efficient, approximate and fast solutions, such as: systems of linear equations, non-linear equations with real or complex roots, numerical integration, systems of non-linear equations or differential equations to name a few, which are included in the curricula of the various engineering courses at least in the fourth semester of their degrees. This position suggests the following research question: What level of proficiency do those students have in prerequisite mathematical topics, such as differential calculus, integration, multivariate calculus, vector calculus, and analytical geometry, when they take higher-level curriculum components? The reasoning above considers that, to successfully face "advanced mathematics," which is part of the curricula of various engineering courses, students are required to demonstrate proper use and management of concepts that should have been learned in the early semesters. However, the reality observed in the classroom manifests an opposite situation that gave rise to this research.

Mateus-Nieves (2023a) states that teachers' daily practice shows the presence of students with low levels of education, who show little development of mathematical competencies, which are key to their professional development as engineers. Given this scenario that seems to be generalised, we note the need to investigate the possible factors that prevent subjects' normal development, such as the aforementioned: advanced mathematics, which, due to its rigorous nature, gets little attention. Furthermore, high school dropout rates resulting

from repeated failures in this type of course are overlooked. That is why this work aims to show, through a mathematical modelling of ten situations typical of engineering education, the difficulties identified during the development of the research and that confirm insufficient education acquired in the prerequisite courses as one of the causes of said school dropout and failure.

We, higher education teachers, must be aware that in the era of globalisation, using and managing computers has mad the teaching and learning processes of higher mathematics easier, on the one side, and complex, on the other because, unconsciously, they have been attributed characteristics that are not intrinsic to them, such as believing that the software will do as told, unaware that if users do not know how to use it, it becomes a mere artefact in their hands. Teaching practice has shown that teachers and students alike have limited digital literacy with computers, electronic devices, or specialised engineering software, whose analysis algorithms already incorporate numerical methods essential in the field of process simulation and provide fast responses when analytical solutions become complex. This situation is exacerbated when students take advanced courses without adequate prior education in higher mathematics. They are also exacerbated when students fail to recognise that numerical analysis studies numerical algorithms to control the error associated with the mathematical approach and obtain a result very close to the exact solution of the problem to be studied. To paraphrase Blomhøj (2004), it is an epistemological precondition that the student can perceive the modelled situation or phenomenon and the mathematics at play as two separate yet interrelated objects. Finally, in the conclusions, this work interprets the problem scenario, considering that the implementation of didactic strategies depends on the teacher, who is responsible for the model, making possible the characterisation and differentiation of mathematical modelling at the research level.

BACKGROUND

Trigueros (2009) states that one way to achieve the contextualisation of knowledge is the presentation of real problem situations in the everyday life of students, feasible to represent through mathematical models that allow answering specific questions in real situations, when decisions are required or when it is imperative to make predictions related to natural and social phenomena. It indicates that mathematical modelling presumes that the argument works as a source of the learning process, since as students work in that context, they are willing to develop mathematical tools and knowledge;

question and investigate situations that provide an opportunity to discuss both the role of these in society, as well as the nature of mathematical models.

Blomhøj (2004) states that mathematics and model analysis are mathematical problems for those who model, and consequently, the modelling process involves solving mathematical problems. Mateus-Nieves and Font (2021) indicate that using mathematical modelling to solve problems in the university educational scenario allows students to engage in learning experiences in which they integrate not only previous knowledge but, through research, deepen, acquire, and propose new learning paths. Hence, we propose a challenge that, in university basic mathematics courses, problem situations be incorporated to reflect the use of concepts and calculations in interpreting solutions, which, in turn, favours the development of mathematical competencies, knowledge, and fundamental values for their professional work.

Mateus-Nieves and Hernández (2021) designed and applied a proposal in which they linked mathematical modelling to the use of software, based on a sequence of ten selected situations typical of engineering training and involving improper integrals. They highlight the importance of modelling as a didactic-dynamic resource because it helps students engage with and understand real-world situations involving improper integrals across different contexts. They conclude that, despite errors detected among students, the strategy showed evidence of the development of mathematical competencies in young people, manifested in the progress in advanced mathematical thinking skills.

Mateus-Nieves (2022a) relates research findings to university students who have difficulty identifying topology as a tool for modelling everyday situations typical of an engineer's work. The author developed and implemented a mathematical modelling proposal in which he used topology to connect the real world with mathematics. In that modelling, the importance of studying topological invariants stands out because they allow us to identify differences and similarities in closed three-dimensional trajectories, which are elements that make up the structure of a surface. It highlights the importance of relating metric space topology and institutionalising a clear, precise, and topology-specific language as a component that allows the student to recognise that the topological properties of the nodes (invariants) are directly related to the properties of the surfaces that can be generated from them. Such elements allowed students to identify that such optimisation can result in a structure with empty spaces where material may be absent and that the modelling enabled by

this geometry could be used as a practical guide to identify areas of traction and compression in the original structure.

Mateus-Nieves (2023b) presents research on engineering students, who showed difficulties relating to and applying trigonometric content when solving problem situations involving trigonometric integration. The study relates to the absence of trigonometric meanings, which prevents them from making connections with the environment and with problem situations typical of their professional education. It shows the design and application of mathematical modelling for three selected problem situations, which allowed students to recognise the importance of trigonometry as a mathematical tool for modelling various problem situations with multiple applications in their professional context. When assessing, the intervention relates to the development of mathematical competencies in young people, manifested in the progress of skills typical of advanced mathematical thinking, including abstracting, generalising, synthesising, defining, and demonstrating**.**

THEORETICAL REFERENCES

This research work is based on two axes: concepts prior to the formalisation of the complex variable function from three references in the course literature: Angulo (2012), Catsigeras (2006), and Pérez (2004), and mathematical modelling. First, it was necessary to review the definitions of some concepts prior to the formalisation of complex-variable functions, including curves or paths, homotopic curves, complex-variable functions, simply connected sets, and the parametrisation of curves, the latter formalised in detail in the results section.

Curves or paths: $\gamma: z = z(t),\ a \leq t \leq b \sqsubset \mathbb{R}$ denotes a parametric curve with parametrisation $z = z(t) = x(t) + iy(t)$ of real variables t, oriented for t ascendency. We will always assume that γ is C^1 piecewise, that is, $z(t)$ is continuous, and except for a finite number of points, there is derivative $\dot{z}(t)$, which is continuous and has finite lateral limits at the points where it does not exist. (Note: $\dot{z}(t)$ is defined as $\dot{x}(t) + i\dot{y}(t)$). The curve is said to $\gamma: z = z(t),\ a \leq t \leq b$ go from point z_1 to point z_2 (or that has a leading end z_1 and a trailing end z_2) if $z(a) = z_1, z(b) = z_2$.

When the object is regularly represented by a function with two variables (x, y), students found it difficult to understand and did not know how to treat it since they did not realise that this type of situation can be simplified when parametric equations are used. Hence, the definition: if γa parametrised curve $\gamma: z = z(t), t \in [a, b]$ is said to be closed if $z(a) = z(b)$; i.e., its initial

end is the same as its final end, where γ^* denotes the set of points of the complex plane traversed by the curve γ, i.e., γ^* is the path of the parametrisation $z = z(t), t \in [a, b]$ of the parametric curve γ. We note that the set γ^* is compact and is denoted by $\gamma \sqsubset \Omega$ when $\gamma^* \sqsubset \Omega$ and $z \in \gamma$ when $z \in \gamma^*$; i.e., when the curve γ passes through the point z. Here, it was necessary to note that given a curve γ, $-\gamma$ is defined as curve γ oriented in the opposite direction. That is, if it $\gamma: z(t), t \in [a, b]$ is oriented toward t, upwards, then $-\gamma: z = z(-\tau), \tau \in [a, b]$ is oriented toward $t = -\tau$, downwards.

We defined: Given two curves γ_1 and γ_2 such that the final end of γ_1 is the initial end of γ_2, we define $\gamma_1 + \gamma_2$ as the curve that is obtained by travelling γ_1 first and then γ_2, in the same side toward which both are oriented. Given two curves γ_1 and γ_2 such that they have the same final end, we define $\gamma_1 - \gamma_2 = \gamma_1 + (-\gamma_2)$ as the curve that is obtained by travelling γ_1 in the same direction in which it is oriented and then γ_2 in the opposite direction to that in which it is oriented.

Homotopic curves: a closed curve γ contained in Ω is homotopic to a point in Ω if there is a continuous deformation of γ that, without leaving Ω, transforms it into a point. That is, there is $z = z_s(t) \in \Omega$ defined for everything $, t \in [a, b]$ and for everything $s \in [0,1]$, which is continuous with respect to (s, t), which for each fixed s, it is a closed curve in Ω, and which, for $s = 0$, it is curve γ, and for $s = 1$, it is a constant curve (equal to a point). That function $z = z_s(t)$ is called homotopy.

Simply connected sets: an open Ω is simply connected if it is connected and any closed curve contained in Ω is homotopic to a point. Two curves γ_1 and γ_2 contained in Ω, both with the same initial end and both with the same final end, are said to be homotopic in Ω if the curve $y_1 - \gamma_2$ is homotopic to a point in Ω.

Complex function: $f: \Omega \to \mathbb{C}$ is a complex function $f = f(z)$ of complex variables $z \in \mathbb{C}$. $u(x, y)$ and $v(x, y)$ denote $Ref(x + iy) = u(x, y)$ y $Imf(x + iy) = v(x, y)$. So $f(z) = u(x, y) + iv(x, y)$ where u_x, u_y, v_x, v_y denote, when they exist, the partial derivatives of u y v in relation to x and y respectively.

Mathematical modelling

An adaptation was made to what was proposed in Mateus-Nieves (2022b) from Blum and Borromeo's (2009, p. 1) standpoint. They define modelling as "the translation process between the real world and mathematics

in both directions", to identify skills associated with this process that should be enhanced in students' education. The development of these skills implies a deeper understanding of reality, strengthens the connections between mathematical concepts, and makes evident the usefulness of mathematics in contextualised situations. To materialise this process, the modelling cycle proposed by Blum and Leiß (2007) was adapted in terms of seven transitions: 1st, students must understand the problem or the proposed situation; that is, they must build a model of the situation (transitions: understanding and construction); 2nd, the situation must be simplified, structured, and precise (transitions: simplification and structuring), giving rise to a real model of it; 3rd, mathematisation, transforms the real model of the situation into a mathematical model; 4th, mathematical work, modelling must produce mathematical results that, 5th, must be interpreted in the real world (transition: interpretation); 6th, validation and verification of results; and here it may be necessary to restart a cycle where repeating some transition(s) is required. 7th student's presentation of the problem and its possible solution. For this, what has already been found and exposed in Mateus-Nieves (2022, p. 760) was considered, who proposes that "an individual can solve a problem, going through different transitions of the cycle without necessarily following the order stated above", and indicates that this modelling cycle may not define a linear thinking path.

METHODOLOGY

This is an exploratory, descriptive, analytical-interpretive work. Of an exploratory nature because it sought to facilitate the understanding of the topic raised; of a descriptive, analytical-interpretive type because it seeks to specify properties and characteristics that students show against the integration of complex variable functions where they had to construct the integral from the parametrisation of the complex curve that determines them; for this, ten problem situations typical of the education of engineers were chosen. A sample of 70 students pursuing various engineering degrees (electrical, mechanical, industrial) at a non-state university was selected based on a written test related to the integration of complex variable functions. Students whose performance was below the expected average (50 points out of 100) were chosen. They were identified from E1 through E70 to exhaustively monitor their work and determine whether the intervention offered any type of improvement in their academic performance, evidenced in the development of an engineer's own mathematical competencies. Each student signed an informed consent form allowing their participation in the research, with their anonymity guaranteed in

the presentation of results. The engineering faculty that coordinates the three programs conducted the research over the course of a school year.

The research was conducted in two phases. First: *readiness*, where the first four transitions are considered. According to the results of the initial test, it was necessary to review and reconstruct previous concepts that students were already expected to handle, but deficiencies were observed when they faced them. For example, correct parametrisation of curves and surfaces, starting with $\mathbb{R}^2, \mathbb{R}^3$ until reaching how to generalise a parametrisation in $\mathbb{R}^n$; ,define what a C^1 piecewise class curve is; when there is a C^1 piecewise class-oriented curve Γ, work exposed in the results section. Second phase: *from formalisation* that includes the three remaining transitions, mathematical modelling was performed on ten proposed situations, of which only some are presented in this report, since they are considered to provide greater relevance to the development of mathematical competencies in students and didactic-methodological elements in teachers. In some cases, it was necessary to sketch conventional and unconventional surfaces using mathematical definitions typical of vector calculus, specifically when parametrising curves and surfaces. Hypermedia was used as a structured set of various media, texts, and graphics as support elements to visualise, analyse, categorise, generalise, and synthesise the proposed situations. The instruments used in this phase were designed, piloted, adjusted, and applied by the team of teachers led by the research director and author of this manuscript. The results were tabulated and triangulated by categories, considering that tests were proposed with exercises where the student had to parametrise a curve in $\mathbb{R}^2$; others in $\mathbb{R}^3$; had to indicate how he does it and what result he finds and whether they had difficulty executing the requested actions.

In another instrument, they were given exercises where they had to find a parametrisation for different solids, including: a sphere of radius *r,* ellipsoid, a circular a cylinder, a parabolic cylinder, a hyperbolic cylinder, a hyperbolic paraboloid, a circular paraboloid, a single- and double-bladed hyperboloids, a cone, and a torus. They were instructed that for the respective answers they should use integrals on paths parametrised by a curve. The information collected allowed for triangulation of data, revealing serious difficulties in handling preconceptions that were expected to be already known by students, and which they apparently did not know or had forgotten. Some of these situations are detailed in the results section. We should note that, due to space, it is not possible to show in detail in this manuscript, two situations related to the parametrisation of surfaces using spherical coordinates, which for the

students represented a greater difficulty. Topic that remains open for further research.

The class sessions were recorded and transcribed into units for analysis; information was drawn from a field journal, where data the researcher observed and considered relevant to the development of the research were recorded. Another instrument used was the creation of analysis units that allowed triangulation with data from the field diary and the results of the applied tests. Such information allowed for detailed monitoring of each student's performance, thereby enabling the drawing of results and conclusions.

ANALYSIS AND RESULTS

The results of the mathematical modelling shared here are focused on identifying two categories: *use and management of integration of complex variable functions* as an application of mathematical concepts typical of engineers' training. The first, *the use*, because engineering students use modelling to design and/or analyse the work of different situations typical of their professional work. The second, to identify the management they give to mathematical concepts to contribute to didactics, particularly of the AMT; to help understand how this mathematical knowledge typical of higher education is articulated.

As a first instrument, a test was designed, piloted, and applied to the population of 135 students. It consisted of ten parametrisations from three different situations; it is reiterated, typical of the engineers' education, focused on answering two central questions: what geometric place does each of the exposed parametrisations represent? and what are the differences between them? The sample selected was 70 students, 55.8% of the population, whose answers were rated as "poor", according to the following criteria:

- *correct* when the answer was complete, indicating direction, starting point, and speed of travel.
- *partially correct*, when any of these aspects were omitted and
- *wrong*, when they do it incorrectly.

The systematisation of results showed that only 28.3% of the population correctly relates previously seen animations to similar parametrisations using the sine and cosine functions. 15.9% partially correct. This allowed us to infer that the sample could not correctly express the analytical expression of the requested parametrisation. Concerned about the high rate of students who present serious weaknesses to perform this type of work; for the *preparation phase*, it was necessary to review the sample on the

correct way to parametrise a curve. They were reminded that there are parametrisations considered "*key*" that must be learned, including: circles and ellipses**.** It was also necessary to review that a parametric equation is a way of *mapping several variables* in terms of a single one. Hence, instead of using coordinates x and y, regularly using coordinates in terms of another variable, here, t were used. For example, when considering $y = x^2$, a Cartesian equation with coordinates (x, y), it is possible to change it, by assuming that $x = t$ and $y = t^2$. From where we have the coordinates (t, t^2) that represent the *parametric shape***.**

For the *parametric equation of a circle*: whose general Cartesian equation is an expression of the form $(x - a)^2 + (y - b)^2 = r^2$, with center $C(a, b)$ and radius r. The first Pythagorean identity is key to square two different values, such as that of the unit circumference, expressed as: $sen^2\theta + cos^2\theta = 1$. A parametrisation of this circumference was shown, taking the aforementioned identity as a strategy to find a way of writing: $9\, sen^2\theta + 9\, cos^2\theta = 9$. We considered that the square of x is equal to: $x^2 = 9\, cos^2 t$means that: $x = 3\, cost$. Similarly, we considered that the square of y is: $y^2 = 9\, sen^2 t$, which means that: $y = 3\, sent$. We found that the parametrisation is in terms of: $x = 3\, cost$ and $y = 3\, sent$.

For the *parametric equation of the ellipse:* its general equation was considered, algebraic expression of the form: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \mathbf{1}$. Where $\boldsymbol{a}$ the distance in direction $\boldsymbol{x}$ (major axis) and $\boldsymbol{b}$ as the distance in direction $\boldsymbol{y}$ (minor axis) are considered. We also showed that it is also possible to parametrise it using the identity used in the circumference. To facilitate understanding, an example was used; they were asked to find an appropriate parametrisation for the ellipse $\left(\frac{x}{9}\right)^2 + \left(\frac{y}{4}\right)^2 = \mathbf{1}$. We proceeded in terms like those used in the circumference. This allowed us to affirm that $\boldsymbol{x^2 = 9\, cos^2 t}$ and $\boldsymbol{y^2 = 4\, sen^2 t}$, meaning that:

$$\left(\frac{x}{9}\right)^2 + \left(\frac{y}{4}\right)^2 = cos^2 t + sen^2 t = 1$$

where we can infer $x = 3\, cost$ and $y = 2\, sent$, that represent *a parametrization* of the ellipse.

Subsequently, it was formalised that a parametrisation of a curve *C* is a vector function $C: I \subset \mathbb{R} \to \mathbb{R}^n$ with the property that, when the parameter varies$t \in I$, its image *C* (*t*) describes the points of *C*. As a strategy, it was

established that a physical interpretation is to think that the parameter *t* represents time and *C(t)* indicates in which position of the plane or space the particle is at the instant *t*. Subsequently, an instrument (guide) was presented, with a series of examples intended to provide ideas and methods for parametrically describing curves; let us look into some of them that offer useful didactic-methodological elements for the teacher who wants to improve their teaching practice.

The software showed a parametrisation for the curve C_1 expressed as:

$$C_1 = \begin{cases} 2x - 3y - z = 2 \\ 2x + y = 1 \end{cases} \quad \text{in } \mathbb{R}^3$$

After graphing, the students were able to identify that C_1 it is a line in $\mathbb{R}^3$, the intersection of the planes $\pi_1: 2x - 3y - z = 2$ and $\pi_2: 2x + y = 1$. By clearing *and* from the equation of π_2, we have $y = 1 - 2x$, by replacing it in the equation $\pi_1: 2x - 3(1 - 2x) - z = 2,$ by simplifying, we reach $4x - z = 5$ by clearing based on *z* we have $z = 4x - 5$, from where the points of C_1 are the (*x*, *y*, *z*) such that $y = 1 - 2x$ and $z = 4x - 5$. This equates $(x, y, z) = (x, 1 - 2x, 4x - 5)$ to:

$$= (0, 1, -5) + (x, -2x, 4x)$$
$$= (0,1, -5) + x(1, -2, 4)$$

that is, C_1 is the line directed by vector $(1, -2, 4)$ that passes through point $(0, 1, -5)$. , which allows us to infer that parametrisation of C_1 is

$$C_1: \mathbb{R} \to \mathbb{R}^3$$
$$C_1(x) = (x, 1 - 2x, 4x - 5)$$

This example made it possible to visualise the way a curve can be parametrised in relation to two planes. However, some students had difficulty identifying the intersection of these planes as a curve in $\mathbb{R}^3$. Students E30, E25, and E67 stated: "*if the intersection is a line, why is it referred to as a curve?*" [transcript analysis unit 289 field diary]. This allows us to infer that for this group, it is not yet clear what type of representation is given in the plane, in space, and even more, in $\mathbb{R}^n$, ignoring that the Differential Geometry course, which they have already taken, considers that space is curved, and in the Vector Calculation course, students are supposed to have already learned to operate vectors in $\mathbb{R}^3$.

We noted that they do not recognise that a *curve* Γ has a direct geometric interpretation as a subset of the plane or space, while a *path* γ is a function that provides a way to parametrise curve Γ, since each point of the curve appears in the form γ(t) for some value of the parameter *t.* This allows us to infer that his mathematical thinking is anchored solely in $\mathbb{R}^2$. This was detailed so that they could overcome this difficulty, and thus, we could show that this void was one of the causes that did not allow them to understand that if $\gamma: [a, b] \to \mathbb{R}^3$ it is a *path* in space, it is possible to write: $\gamma(t) = (x(t), y(t), z(t)) = x(t)i + y(t)j + z(t)k$, with $a \leq t \leq b$, where x, y, z are continuous functions of $[a, b]$ in $\mathbb{R}$. Therefore, the *path*: γ has parametric equations:

$$\begin{cases} x = x(t) \\ y = t(t) \\ z = z(t) \end{cases} \qquad a \leq t \leq b \quad \triangleq$$

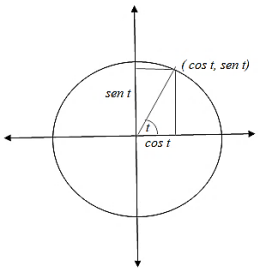
Another situation to highlight was when students were asked to determine whether the function expressed as:

$$C_2: [0, 2\pi] \to \mathbb{R}^2$$
$$C_2(t) = (cost, sent)$$

represents a parametrisation of circumference C_2 outlined in Figure 1, whose equation is $x^2 + y^2 = 1$ in $\mathbb{R}^2$

Figure 1

Parametric representation of the circumference.

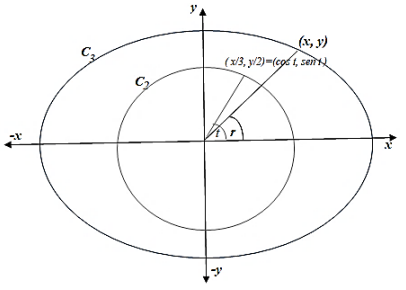


When exploring the situation with the students, it was observed that students looked at *t* as the parameter that represents a certain time; therefore, it was necessary to explain in detail that, for this example, the parameter *t*

represents the angle formed by the vector of origin (0, 0) and end (*x, y*) with the positive semi-axis of the abscissa. This allowed us to identify shortcomings with the use and management of vectors in $\mathbb{R}^2$. As a strategy to overcome this difficulty, an ellipse was interposed on $\mathbb{R}^2$, where t no longer has the same interpretation as in the circumference, that is: Let $C_3: \frac{x^2}{9} + \frac{y^2}{4} = 1$ be represented in Figure 2 with the condition that the graph of the ellipse is on the same plane as the circumference of radius 1, share the center, and that it is on the origin of the plane.

Figure 2

Representation: a parametrisation of the ellipse on a circumference.



Such elements allowed them to identify that a parametrisation of this ellipse whose equation can be written in the form: $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$, means that a point $(x, y) \in C_3$ if and only if $\left(\frac{x}{3}, \frac{y}{2}\right) \in C_3$. They were guided to consider that, if this occurs, we have $\frac{x}{3} = \cos t$ and $\frac{y}{2} = sent$ for $t \in [0, 2\pi]$. In such a way that by clearing x and y, we obtain a parametrisation of this ellipse, which is obviously the function expressed as:

$$C_3: [0, 2\pi] \to \mathbb{R}^2$$

$$C_3(t) = (3\cos t, 2\ sen\ t)$$

Once the difficulty in $\mathbb{R}^2$ was overcome, they could formalise that a parametrised class curve C_1 in $\mathbb{R}^n$ *is an application* $\gamma: [a, b] \to \mathbb{R}^n$ of class C_1, where $a, b \in \mathbb{R}$ with $a < b$, such as $\gamma'(t) \neq 0, \forall\ t \in [a, b]$. Here, another difficulty arose in correctly defining the vector tangent $\gamma'(t)$ to the curve γ at each point γ(t). Since for $t = a$ they are supposed to exist and their limits are

equal, i.e.: $\gamma'_+(a) = \lim_{t\to a^+} \frac{\gamma(t)-\gamma(a)}{t-a} = \lim_{t\to a^+} \gamma'(t)$, analogously, it is also given in $t = b$. For this sample, the situation was not easy to understand, given that, despite having already taken and succeeded in the Vector Calculation course and that the use of the software allowed them to visualise the existence and equality of the limits above, they did not find it so easy to understand and apply.

Now, in the formalisation phase, students were shown that it is important to recognise what is usually said: the circumference $C(o,r) = \{z: |z| = r\}, r > 0\ fijo$ is a curve, when it is the *trace of the curve* and given by the function $\gamma(t) = re^{it} = rcosit + irsent \cong (rcost, rsent), t \in [0, 2\pi]$. In other words, what you actually have is that $C(o,r) = \{\gamma\}$. This, without specifying function γ, since we do not know how the trace is traced in time. These elements allowed them to identify that $\gamma_1(t) = re^{-it} = r\cos t - irsent \cong (rcost, -rsent)$, with $t \in [0, 2\pi]$ represents another different curve, because it is not the same function; but that it has the same trace as $\gamma: C(0,r) = \{\gamma_1\}$. In this way, they identified that function γ describes the movement, with uniform speed, along the circumference of radius *r* centred at the origin in the positive direction, while γ_1 describes the movement, also with uniform speed, along the same circumference, but in the negative direction, that of the clock hands.

In this part of the research, the software was fundamental. It served to overcome the difficulties described above, enabling them to formalise and generalise that a parametrised curve $\gamma: [a,b] \to \mathbb{R}^n$ is equivalent to another parametrised curve $\sigma: [c,d] \to \mathbb{R}^n$ if there is a bijection $\phi: [c,d] \to [a,b]$ of class C^1 such that $\phi'(t) > 0$ and $\sigma(s) = \gamma(\phi(s))$ for every $s \in [c,d]$. This immediately verifies that the ratio $\gamma \Re \sigma \leftrightarrow \gamma$ is equivalent to σ; and it is effectively an equivalence ratio in the set X of parametrised curves of class C^1in $\mathbb{R}^n$. Understanding this, the *quotient set* X/ was formalised $\mathbb{R}$ as the set of oriented curves of class C^1in $\mathbb{R}^n$.

What has been advanced so far has served the sample to learn to estimate the value of the integral along a curve, a basic skill that will be essential when advancing demonstrations and solutions to proposed situations. It was necessary to clarify the distinction between parametrisations (paths), curves, and traces, review basic definitions and properties, so that students would get used to the complex notation, which was new to them.

It was necessary to define that each oriented curve Γ = [γ] of class C^1 allows us to observe how the trace $\gamma[a,b]$ of a parametrised curve $\gamma: [a,b] \to \mathbb{R}^n$ of class C^1 represents the equivalence class [γ], with the orientation induced by γ. Here, we had to observe that this same trace $\gamma[a,b]$ admits two orientations:

one given by γ, which runs $\gamma[a,b]$starting at $\gamma(a)$ and ending at $\gamma(b)$, and another given by the curve opposite to γ, defined by $\gamma^{-}:[a,b]\to\mathbb{R}^n$ where $\gamma^{-}(t)=\gamma(a+b,t)$, which runs $\gamma[a,b]$ starting at $\gamma(b)$ and ending at $\gamma(a)$. The oriented curve of class C^1 represented by this opposite curve γ^{-} was denoted $\Gamma^{-}=[\gamma^{-}]$. Similarly, it was possible to define *piecewise curves of class* C^1. It was possible to say that a continuous application $\gamma:[a,b]\to\mathbb{R}^n$ is a parametrised piecewise curve of class C^1 if there is a $a=a_0<a_1<\cdots<a_m=b$, a partition of $[a,b]$ such that the restriction of γ to each sub-interval $[a_j,\ a_{j+1}]$, with $j=0,1,\cdots,m-1$, defines a parametrised curve of class C^1 where γ admits a single tangent vector $\gamma'(t)$ at each point $\gamma(t)$ with $t\neq a_j$ for all j, while at points $\gamma(a_j)$ they have a tangent to the left and another tangent to the right of $\gamma(a_j)$, respectively defined by vectors $\gamma_{-}(a_j)=\lim_{t\to a_j-}\frac{\gamma(t)-\gamma(a_j)}{t-a_j}$ and $\gamma_{+}(a_j)=\lim_{t\to a_j-}\frac{\gamma(t)-\gamma(a_j)}{t-a_j}$ possibly different.

The situation above allowed students to understand that a parametrised curve $\gamma:[a,b]\to\mathbb{R}^n$ of a piecewise class C^1 is equivalent to another parametrised curve $\sigma:[a,b]\to\mathbb{R}^n$ of a piecewise class C^1 if there are partitions $a=a_0<a_1<\cdots<a_m=b$ of $[a,b]$ and $c=c_0<c_1<\cdots<c_m=d$ of $[c,d]$ such that restriction $\gamma|_{[a_j,a_{j+1}]}$ is a parametrised curve of class C^1 equivalent to the restriction $\sigma|_{[c_j,c_{j+1}]}$, for every $j=0,1,\cdots,n-1.$. This allowed them to observe that it is a relationship $\gamma\,\Re\,\sigma\leftrightarrow\gamma$ equivalent to σ in the set of parametrised curves of piecewise class C^1 in $\mathbb{R}^n$.

Subsequently, the resulting quotient set was formalized as: *set of oriented curves of piecewise class* C^1 *in* $\mathbb{R}^n$, which allowed us to define, generally, that an oriented curve Γ of piecewise class C^1 represented by a parametrised curve $\gamma:[a,b]\to\mathbb{R}^n$, allows us to define Γ^{-} as the equivalence class represented by $\gamma^{-}(t)=\gamma(a+b,t),t\in[a,b]$. Where, in each parametrised curve $\gamma:[a,b]\to\mathbb{R}^n$ of class C^1 piecewise function $\|\gamma'(t)\|$ is integrable in $[a,b]$, being well defined except for a finite number $a_0,a_1,\cdots a_m$ of points of $[a,b]$, and being continuous and bounded in $[a,b]\setminus\{a_0,a_1,\cdots a_m\}$. Defining the length of γ by $long(\gamma)=\int_a^b\|\gamma'(t)\|dt$.

When using the variable change theorem, students could note that if γ and σ are (parametrised curves of class C^1) equivalent then $long(\gamma)=long(\sigma)$, which $long(\gamma)=long(\gamma^{-})$; therefore, given an oriented curve Γ piecewise class C^1 allows to define $long(\Gamma)=long(\gamma)$, where γ is any parametrisation of said curve. They could observe that the curve Γ is the class of all curves equivalent

to γ. Here it is important to highlight the analysis unit 487 which indicates that one of the subgroups, E24, E13, and E35, correctly identified that [*an oriented parametrisation Γ class* C^1 *piecewise in* $\mathbb{R}^n$ *has three options:*

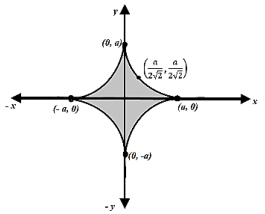
- A closed curve in $\mathbb{R}^n$ admits a parametrisation $\gamma: [a, b] \to \mathbb{R}^n$ such that $\gamma(a) = \gamma(b)$.
- A simple curve supports a parametrisation $\gamma: [a, b] \to \mathbb{R}^n$, that is injective;
- A simple closed curve supports a parametrisation $\gamma: [a, b] \to \mathbb{R}^n$such that $\gamma(a) = \gamma(b)$ and the restriction of γ to the interval $[a, b)$ is injective].

This allows us to infer that this part of the sample (students) clearly identified when a curve is rectifiable and smooth piecewise. They were shown that even with the hypothesis $\gamma \in C[a, b]$ one cannot always get what intuitively corresponds to the notion of a curve; for example, there are curves such that $\gamma: [a, b]$ is a square or other similar sets, commonly called Peano curves. We stressed that usually more conditions are requested that a curve has to meet; among them, that γ has a derivative at all points of $[a, b]$, except for a finite set of points, although that does not avoid all possible problems, because it is feasible that the derivative may not be bounded near some points. The importance of observing that, when considering two equivalent parametrisations, γ and Γ of the same curve that is either smooth or piecewise, it is always possible C^1 to assume that the function α, which represents a parameter change, is also a smooth or piecewise C^1 function. This made it easy for them to see that the length is the same for any other parametrisation of a smooth curve; in fact, they noticed that the formula reached the same shape after a monotonous change of variable. Here, it was formalised that the word *contour* is frequently used to refer to a simple, closed, piecewise curve C^1.

Another situation that was proposed to the students and correctly performed was to calculate the area of the region enclosed by the hypocycloid defined by $x^{2/3} + y^{2/3} = a^{2/3}$, using the parametrisation $x = acos^3\theta$, $y = sen^3\theta$, $0 \leq \theta \leq 2\pi$, Figure 3 was offered as a guide

Figure 3

Guide chart of the hypocycloid.



Delivering the parametrisation and graph of the situation helped students understand how the exercise should be performed. It was easier for them to understand the situation and identify which type of double integral to use. However, this did not happen in situations where they had to find parametrisations; we observed that they find it difficult to build them because they do not handle preconceptions that they should already know because they were studied in previous courses. The analysis unit 456 is presented, describing the product delivered by two subgroups that identified the area A defined by:

$$A = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx \quad \text{equivalent to}$$

$$\frac{1}{2}\int_0^{2\pi}[(a\,cos^3\theta)(3a\,sen^2\theta\cos\theta) - (a\,sen^3\theta)(-3a\,cos^2\theta\,sen\,\theta)]\,d\theta$$

$$= \frac{3}{2}a^2\int_0^{2\pi}(sen^2\theta\,cos^4\theta + cos^2\theta sen^4\theta)\,d\theta = \frac{3}{2}a^2\int_0^{2\pi} sen^2\theta\,cos^2\theta\;\,d\theta$$

$$= \frac{3}{8}a^2\int_0^{2\pi} sen^2\,2\theta\,d\theta = \frac{3}{8}a^2\int_0^{2\pi}\left(\frac{1-\cos 4\theta}{2}\right)d\theta$$

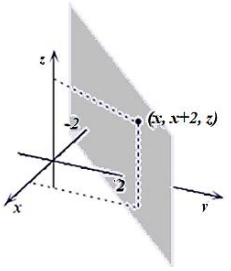
$$= \frac{3}{16}a^2\int_0^{2\pi} d\theta - \frac{3}{16}a^2\int_0^{2\pi}\cos 4\theta d\theta = \frac{3}{8}\pi a^2\,. \;\triangleq \text{[analysis unit 456]}$$

When formalising a parametrisation for surfaces $S \subset \mathbb{R}^3$, we noted students found it difficult to identify that the variation of the parameters $s \in [a, b]$ and $t \in [c, d]$, their image $\phi(x, t)$ describes the points of surface *S*. In this situation, it was evident that students cannot relate that the surface in $S \subset \mathbb{R}^3$ is an applicationϕ: $[a, b]$ x $[c, d] \rightarrow \mathbb{R}^3$. This led the work to the need to formalise:

- Sea $S: y - x = 2$ represents a vertical plane. The points of S are characterized by having their first two coordinates related by $y = x + 2$, while the last coordinate can take any value; see Figure 4.

Figure 4

Representation of the surface of a vertical plane in space $\mathbb{R}^3$

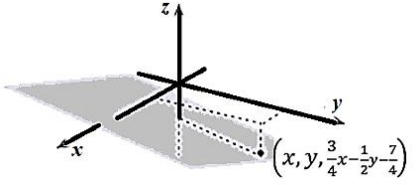


then, the application $\phi: \mathbb{R}^2 \to \mathbb{R}^3$ given by $\phi(x, z) = (x,\ x + 2, z)$ is a parametrisation del plane *S*.

- If *S* represents a non-vertical plane, for example $S: 3x - 2y + 4z = 1$, the coefficient of z will always be non-zero, which allows z to be cleared from the equation that defines S in the following terms: $z = \frac{3x}{4} - \frac{y}{2} - \frac{7}{4}$ (Figure 5).

Figure 5

Representation of the surface of a horizontal plane in space $\mathbb{R}^3$



in which it is possible to deduce that a point $(x, y, z) \in S$ if and only if $(x, y, z) = \left(x, y, \frac{3x}{4} - \frac{y}{2} - \frac{7}{4}\right)$ where x, y are any real numbers. Then, the application $\phi: \mathbb{R}^2 \to \mathbb{R}^3$ given by $\phi(x, y) = \left(x, y, \frac{3x}{4} - \frac{y}{2} - \frac{7}{4}\right)$ represents a parametrisation of *S*.

As complementary and practical exercises, we proposed that they found a parametrisation for different solids: a sphere of radius r, an ellipsoid, a circular cylinder, a parabolic cylinder, a hyperbolic cylinder, a hyperbolic paraboloid, a circular paraboloid, a single-bladed and a double-bladed hyperboloid, a cone, and a torus. Calculation was also requested $\int_\gamma f(z)dz$ for the following cases:

a. $f(z) = \overline{z}$ with γ the circumference of centre 0 and radius 2 positively oriented;
b. $f(z) = \mathcal{R}z$ with γ the unitary semicircle passing through $-i, 1, i$ in that order;
c. $f(z) = \frac{1}{z}$ with γ the polygon joining the points $1, -i, -1, i$ in that order.

The capacity achieved by several subgroups of students in the sample is highlighted, who, in several of their answers, used integrals on paths parametrised by a curve. Let us see the proposal presented by the subgroup of students E39, E45, and E68, manifested in the analysis unit 524 in the following terms:

If $\gamma: [a, b] \to C$ is a parametrised path and *f(z)* a continuous complex function over the path trajectory, where the integral of the function *f* along the path γ is

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

it is sufficient to parametrise the three paths that are proposed. For the 1st, a parametrisation would be $\gamma(t) = 2e^{it}$ with $t \in [0, 2\pi]$, therefore

$$\int_\gamma f(z)dz = \int_0^{2\pi} 2e^{-it}\, 2ie^{it}\, dt = 4i\int_0^{2\pi} dt = 8\pi i.$$

In the 2nd, the parametrisation $\gamma(t) = e^{it}$ with $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ since $\mathcal{R}(e^{it}) = \mathcal{R}\,(\cos t + isent) = \cos t = \frac{e^{it}+e^{-it}}{2}$. . In this case, we have the parametrisation $\gamma(t) = e^{it}$ with $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and as $\mathcal{R}(e^{it}) = \mathcal{R}\,(\cos t + isent) = \cos t = \frac{e^{it}+e^{-it}}{2}$ we obtain:

$$\int_{\gamma} f(z)dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{it}+e^{-it}}{2} ie^{it}dt = i\frac{\pi}{2}.$$

For the 3rd, we can say that it has a slightly different shape from the previous two. Because a branch cut of the complex plane must be chosen so that the polygonal γ is in the complement of said cut. For this, we consider r as the half-line whose origin is at 0 and forms an angle of $\frac{\pi}{4}$ with the real axis. In $C \setminus r$, the function $f = \frac{1}{z}$ has an antiderivative: it is $l(z) = \log|z| + i\theta$; being $z = |z|e^{i\theta}$ with $\frac{\pi}{4} < \theta < \frac{\pi}{4}+2\pi$. From which:

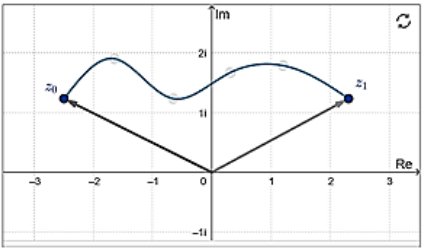
$$\int_{\gamma} f = l(i) - l(1) = \left(0 + i\tfrac{\pi}{2}\right) - (0 + 2\pi i) = \frac{3\pi i}{2}.$$

What is contradiction $\triangleq$. [transcript analysis unit 524]

When validating the work carried out by these two subgroups before the plenary of the sample, we observed that it was difficult for some of their colleagues to understand, since they did not have the correct way to parametrise a curve clearly; therefore, it was necessary to reinforce through assuming that the functions of real value $x = x(t)$, and $y = y(t)$, with $a \leq t \leq b$ are parametric equations of a curve C in the complex plane. And that by considering these equations as the real and imaginary parts in $z = x + iy$, it allows us to describe the points z in C by means of the complex value function of a real variable t. This helped formalise a *parametrisation* of C in the following terms: $z(t) = x(t) + iy(t)$, with $a \leq t \leq b$. Where point $z(a) = x(a) + iy(a)$, or point $z_0 = (x(a), y(a))$ called as the *starting point* of C and$z(b) = x(b) + iy(b)$, or $z_1 = (x(b), y(b))$ as the *end point*. Some students could identify that the $z(t) = x(t) + iy(t)$expression can be interpreted as a two-dimensional vector function. And in an analogous way, that $z(a)$ and $z(b)$ it is possible to identify them as the position vectors. So, when t ranges from $t = a$ to $t = b$, the curve Cdrawn by the movement of the tip of the vector $z(t)$. can be observed. Figure 6 shows the use of an online GeoGebra applet, where the variation is considered $-2 \leq t \leq 1$.

Figure 6

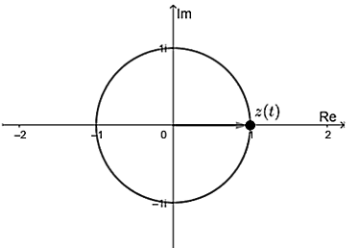
Visualisation in the software of the start and end point in a parametrisation



At this point in the research, we could observe that more than half of the sample managed to identify that the parametric equations $x = \cos t, y = sent$ with $0 \leq t \leq 2\pi$ describe a circle of unit radius centred at the origin. And that a parametrisation of this circle is $z(t) = \cos t + isent$, or $z(t) = e^{it}$ with $0 \leq t \leq 2\pi$. Figure 7

Figure 7

Parametrisation of a circle in the complex plane



Thus, they could identify that there is no single parametrisation for a given contour C, which allowed the following expressions to be verified:

1. $z(t) = e^{it} = \cos t + i\ sen\ t,\ \ 0 \leq t \leq 2\pi$
2. $z(t) = e^{2\pi it} = \cos(2\pi t) + i\ sen\ (2\pi t),\ \ 0 \leq t \leq 1$
3. $z(t) = e^{\frac{\pi}{2}it} = \cos\left(\frac{\pi t}{2}\right) + i\ sen\ \left(\frac{\pi t}{2}\right),\ \ 0 \leq t \leq 4$

recognizing that they are positively oriented parametrisations for the unit circle $|z| = 1$. This allows us to infer that they managed to identify positive orientation and differentiate it from negative. Here, the use of the software was key, as it allowed them to identify and distinguish that the parametrisations presented respond to routes in different directions and with different starting

points, also with different speeds. For example, it was possible for them to determine that if it is a particle that travels around the circumference, it is possible to find two paths, one positive, one negative, and ask: what are the differences between these paths? Does anything change? What changes? Our aim is this work was that students related the animations previously seen to generalise to similar parametrisations using the sine and cosine functions. They were presented with situations verbally for them to present them analytically. Students have the software to verify if their answer is according to the request, for example, to the question: What should be the correct parametrisation if the particle moves counterclockwise and moves 3 times faster than curve 1, starting its trajectory at point $(1, 0)$? We noted that when the equations appear explicitly, it makes their work easier, whereas when they appear implicitly, they find it difficult to build them. Several subgroups of students answered the question only by repeatedly watching the corresponding video that had been offered to them to pose the corresponding system. Only 65% of the sample did it correctly. This allows us to infer the implementation and proper use of hypermedia[1], which allowed them to achieve advances in the construction of cognitive processes evidenced in the development of mathematical competencies. It is important to mention that in the rest of the sample, there was medium progress when presenting the equations separately and considering the parameter that verifies them both.

CONCLUSIONS

The elements shown in this study allow us to recommend the non-promotion of students with low academic education in courses considered prerequisite; particularly those in which there is little use and management of concepts that are expected to be known and learned in previous courses. Promoting them with these shortcomings causes difficulties such as those manifested here and delay when it is intended to develop topics of subsequent courses that have as a prerequisite the proper use and management of concepts considered key and were worked on in previous courses.

Before parametrising complex curves, the knowledge of the students when they begin the course must be diagnosed to identify shortcomings and on them plan reviews and formalise the correct way to do them in $\mathbb{R}^2, \cdots, \mathbb{R}^n$. Only in this way can they understand that a curve can be represented with different

[1] Conceptually, *hypermedia* designates means that the individual can bifurcate or execute

parametrisations and that it is possible to take any of the equivalent paths that compose it as a representative. Here, I emphasize, we have an "abuse of terminology" in which teachers sometimes incur when saying that a continuous function $\gamma\colon [a,b] \to \mathbb{C}$ is a curve, instead of saying that it is a parametrisation (or path) that represents the curve, and that it is its equivalence class. If the student understands that choosing one parametrisation instead of another is equivalent to reparametrising the curve. The student will be able to understand the *function α as a parameter change*. This would allow them to identify the context when we refer to a curve and when we refer to a parametrisation. I have observed that the term *path* is regularly used as a synonym for *curve,* elements that, if not clear, can reinforce the difficulties presented in students.

When the trace of a curve was referred to as set $\{\gamma\} = \gamma([a,b])$, for any parametrisation $\gamma\colon [a,b] \to \mathbb{C}$ where point $\gamma(a)$ is called the start point of the curve, and $\gamma(b)$, the end point, we observed that in some textbooks, terminology is also abused, using *curve* when actually referring to *trace*. These elements are key for the teacher in selecting the course bibliography.

The integration of technology into the curriculum is considered of utmost importance. Here, the planned and applied hypermedia were adequate for students with as many identified difficulties as the sample described here, to distinguish the effects produced by different parametrisations of the same geometric place. The negative of this part was the amount of extracurricular time that was necessary for these young people to advance in their formative process. This situation is not always possible to execute.

We recommend that, when planning the hypermedia, professors consider the population to which it is directed to create didactic material tailored to their needs, supported by videos, infographics, text, images, and specialised software, elements that make the teaching process progressive and agile. Although we are in the age of global digitisation, difficulties were observed in most of the sample in managing these resources. This suggests that they still do not obtain the maximum benefit, perhaps because they have not yet acquired study techniques appropriate to their level of study; some seem to be in the process of adapting to university life. Therefore, we recommend not providing too much information, but using different formats, verbal, analytical, videos, graphics, and organising adequate hyperlinking for organisation.

In this work, incorporating animations from online GeoGebra applets was crucial because it allowed them to clearly see how the path of a curve changes with its parameter. This facilitated his comprehension, as opposed to studying the topic solely from printed books or using the board in the

classroom. However, the technological tool by itself does not improve either the learning or the teaching process. Therefore, the teacher plays a fundamental role as a guide in its use; the teacher is the first to assist and stimulate exploration and visualisation and indicate different options for using hypermedia resources to solve the problems posed.

AUTHORS' CONTRIBUTION

EMN led and developed the research project and the organisation of this manuscript.

DATA AVAILABILITY DECLARATION

The data supporting this study will be made available by the corresponding author (EMN) upon reasonable request.

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