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Assessment of algebraic reasoning of students in secondary education

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ABSTRACT

Background: Various investigations highlight the importance of developing algebraic reasoning from the earliest educational levels and extending it throughout academic education**. Objectives**: We evaluated students' algebraic reasoning in compulsory secondary education as they solved problems involving patterns, functions, equations, and systems of equations. **Design**: We adopted a methodological approach that combines qualitative and quantitative analyses. We analysed students' solutions to four problems, one for each of the previously mentioned dimensions of school algebra. **Setting and participants**: 235 students of the first three years of secondary education (12–15 years) from public schools in the provinces of Almería, Granada, and Jaén (Spain) participated. **Data collection and analysis**: A content analysis of students' productions and a statistical treatment of the data were conducted to examine trends in the development of algebraic reasoning and the degree of correction across school grades. **Results**: Although performance varied by problem, students had difficulty solving the tasks, most often using pre-algebraic strategies. There was also a progressive increase in students' algebraic reasoning throughout the school year, along with a positive relationship between the level of algebraic reasoning and the correctness of answers, especially in problems involving algebraic functions and equations. **Conclusions**: Our findings underscore the importance of designing tasks that promote the explicit use of algebraic reasoning from the earliest grades.

Keywords: Algebraic Reasoning; Compulsory Secondary Education; Patterns; Functions; Equations.

Avaliação do raciocínio algébrico de alunos do ensino secundário obrigatório

RESUMO

Contexto: Diversas investigações apontam para a importância de desenvolver o raciocínio algébrico desde os primeiros níveis educativos e prolongá-lo ao longo da formação académica. **Objetivos**: Avaliamos o raciocínio algébrico demonstrado por alunos do Ensino Secundário Obrigatório na resolução de problemas que envolvem padrões, funções, equações e sistemas de equações. **Design**: Adotamos uma abordagem

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metodológica que combina análise qualitativa e quantitativa. Analisamos as resoluções dos alunos para quatro problemas, um para cada dimensão da álgebra escolar mencionada anteriormente. **Ambiente e participantes**: Participaram 235 alunos dos três primeiros anos do ensino secundário (12–15 anos) provenientes de escolas públicas das províncias de Almería, Granada e Jaén (Espanha). **Coleta e análise de dados**: Foi realizada uma análise de conteúdo das produções dos alunos e um tratamento estatístico dos dados para examinar tendências no desenvolvimento do raciocínio algébrico e o grau de correção em função do ano letivo. **Resultados:** Os resultados mostraram que, embora existissem diferenças de desempenho em função do problema tratado, os alunos tiveram dificuldades em resolver com sucesso as tarefas, utilizando com maior frequência estratégias de caráter pré-algébrico. Observou-se também um aumento progressivo do raciocínio algébrico utilizado pelos alunos à medida que o ano letivo avançava, bem como uma relação positiva entre o nível de raciocínio algébrico e o grau de correção das respostas, especialmente em problemas que envolvem funções e equações do tipo algébrico. **Conclusões**: Nossas descobertas destacam a importância de criar tarefas que promovam o uso explícito do raciocínio algébrico desde os primeiros anos letivos.

Palavras-chave: raciocínio algébrico; ensino secundário obrigatório; padrões; funções; equações.

INTRODUCTION

The teaching of algebra is a key aspect of mathematics education, as it provides the foundation for higher-level mathematical content and is essential in many STEM fields (Veith et al., 2023). Various theoretical perspectives recommend attending to algebraic reasoning from the earliest educational levels to strengthen mathematical activity in schools and facilitate the transition to high school mathematics (Godino et al., 2014; Kieran, 2022; Radford, 2018). Recent curriculum proposals have echoed this concern (ACARA, 2014; CCSSI, 2015; MEFP, 2022a). Thus, algebraic reasoning appears in primary education curricula through the recognition of dependence relationships between variables, the expression of generalities, the modelling of different phenomena, and the manipulation of symbolic representations involved (ACARA, 2014; CCSSI, 2015; MEFP, 2022a). This multidimensional perspective continues to define algebra in secondary education, so that, in the Spanish curriculum, the algebraic sense contemplates the study of patterns and regularities, the determination of the formation rule in simple cases, the study and symbolic expression of linear relationships, the solution of equations and linear systems, the concept of variable, and the study and comparison of linear relationships through different modes of representation (MEFP, 2022b).

Although there is research focused specifically on intermediate levels of secondary education (Durán Salas, 2023; Paoletti et al., 2021; Papadopoulos

& Thoma, 2023; Pitta-Pantazi et al., 2020; Utami et al., 2023) in recent decades, there has been a shift in research towards early algebra (Ellis & Özgür, 2024). Despite the importance of characterising the different dimensions of algebraic thinking (Pitta-Pantazi et al., 2020) and recognising how secondary school (high school) students progress (Akkan, 2013; Chvál et al., 2021; Ellis & Özgür, 2024; Muñoz et al., 2025; Paoletti et al., 2021; Pitta-Pantazi et al., 2020; Utami et al., 2023; Zwanch, 2022), few studies in the Spanish context characterise the knowledge and difficulties of secondary school students in solving algebraic tasks (Ruano et al., 2008; Socas et al., 2016).

With this interest, our goal is to evaluate students' algebraic reasoning in the first three years of compulsory secondary education (CSE) as they solve problems involving patterns, functions, equations, and systems of equations. Following the model proposed by Godino et al. (2014), recently expanded by Burgos et al. (2024), we understand the algebraic character of students' mathematical practices in terms of the presence of certain objects and processes, i.e., types of representations used, the different degrees of generalisation and functional reasoning, and the analytical calculation involved. This model provides a series of criteria that allow us to identify purely arithmetic practices and distinguish them from successive levels of algebraisation. The levels of algebraic reasoning do not describe the subjects' cognitive profiles; that is, they are not indicative of stages of cognitive development prior to a class of mathematical tasks. The application of the levels of algebraic reasoning to practice systems allows us to characterise and distinguish categories of algebraic reasoning from both institutional and personal perspectives on algebra, not limited to the context of application (Vergel et al., 2023).

We posed the following research questions:

- What levels of algebraic reasoning do students in different secondary school levels mobilise when solving problems?
- Is there a relationship between the level of algebraic reasoning and the degree of correctness in solving the problems?
- Does the type of problem (patterns, functions, equations, systems of equations) influence the levels of algebraic reasoning or the degree of correction achieved?

BACKGROUND

Multiple studies show students' ability to think in algebraic terms from primary education (Ayala-Altamirano & Molina, 2021; Blanton et al., 2015;

Kieran, 2022; Kieran & Martínez-Hernández, 2022; Radford, 2018). Students can reason relationally, recognise relationships between the terms of expressions, and use fundamental properties of operations to transform them (Wilkie & Hopkins, 2024), recognise functional relationships and justify them (Blanton et al., 2015; Brizuela & Earnest, 2008; Ayala-Altamirano & Molina, 2021; Moreno et al., 2016), and recognise patterns and determine the general rule that allows them to be built (Hunter & Miller, 2022; Mulligan et al., 2020). These investigations also highlight the importance of teacher intervention in the early development of algebra. However, "many attempts to better prepare students for algebra have not resulted in improved levels of academic achievement" (Egodawatte, 2023, p. 30), and secondary and even high school students still struggle with algebraic concepts and skills (Egodawatte, 2023).

According to Pitta-Pantazi et al. (2020), secondary school students' algebraic thinking is composed of four dimensions: functional thinking, generalised arithmetic, modelling, and algebraic demonstration. The authors consider that although algebraic thinking can be incorporated into the mathematics curriculum from the earliest levels, it is important to consider which types of algebraic thinking can be developed first and which require a prior foundation. Thus, their research shows that high school students are able, in the first place, to perform functional thinking tasks, such as identifying patterns and finding functional relationships. They can then develop skills related to generalised arithmetic, such as identifying the structure and general properties of expressions, treating numbers algebraically, and working with equalities and inequalities. This progressive development allows us to advance in the construction of models, such as equations, to mathematise specific situations. Finally, these skills provide students with the basis for making arguments without reference to specific quantities, through mathematical processes such as generalisation and syntactic calculation (Pitta-Pantazi et al., 2020).

Previous research has focused on characterising secondary school students' algebraic thinking across these dimensions, especially in the functional context (Ellis & Özgür, 2024). For example, Utami et al. (2023) analysed the functional thinking of high school students (aged 13–14) using patterns. They observed that most students identified patterns using a recursive approach; however, they had difficulty expressing the general rule symbolically. As in previous work (Wilkie, 2016), students who did not have functional thinking at the correspondence level could not correctly produce algebraic expressions of pattern generalisations. In addition, these students also showed greater difficulty in generating patterns in word problems. Similarly, Akkan (2013) observed, in a comparative study of primary and secondary students (aged 10–15), that the effectiveness and variety of strategies for generalising patterns increased with students' level of education, although recursive strategies remained dominant.

Other research has focused on analysing how high school students approach tasks that involve modelling situations with equations and inequalities, or solving them. In an investigation with students aged 12 and 13, Chvál et al. (2021) found that half of the participants made mistakes when solving first-degree linear equations, highlighting the main errors as not applying the hierarchy of operations correctly or difficulties with clearing. Van Amerom (2003) observed that primary and secondary students had problems modelling contextualised situations using literal symbols to formulate equations. In this sense, research such as that by Johanning (2004) or Zwanch (2022) indicates that secondary school students tend to use trial-and-error arithmetic strategies to solve contextualised problems that can be modelled using equations or systems of equations. Most research examining the meanings secondary school or university students attribute to linear inequalities shows that they apply the same algebraic techniques to solve these inequalities as they do to solve equations (Muñoz et al., 2025; Paoletti et al., 2021). Students see the sign "greater than" or "less than" as a link between two algebraic expressions, so even when they correctly solve the inequality, they do not adequately interpret the result or reflect on the coherence of the solution with the conditions of the problem. In other cases, they do not conceive of the solution as a range of values, restricting themselves to a single numerical value (Paoletti et al., 2021). For authors such as Muñoz et al. (2025), these difficulties are largely explained by an approach to teaching that prioritises syntactic calculation without considering its articulation with other representation registers, such as the graph.

ELEMENTARY ALGEBRAIC REASONING LEVELS MODEL

Adopting the pragmatic perspective of the onto-semiotic approach (OSA) (Godino, 2024), elementary algebraic reasoning (EAR) is conceived as a system of operational and discursive practices involved in the resolution of tasks accessible in primary education that involve algebraic objects and processes (Godino et al., 2014). Types of algebraic objects are considered: a) *binary relationships* of equivalence or order and their respective properties; b) *operations and their properties*, performed on the elements of diverse sets; c) *functions*, their operations and properties; d) *structures* (semigroup, group, etc.), their types and properties. These objects emerge from practices through processes of *generalisation* (identification of intensive or general objects, i.e., classes of extensive or particular objects), *unitisation* (the intensive becomes something new, different from the elements that constitute it, a unitary entity), *representation* (materialisation of the unitary entity), and *syntactic calculation*. Godino et al. (2014) propose an EAR model that allows us to characterise the level of algebraic reasoning involved in a mathematical activity. Recently, Burgos et al. (2024) extended the EAR model of Godino et al. (2014) by considering different degrees of sophistication in essential features of algebraic reasoning:

- *Generalisation*. Generalisation as a process implies: identifying elements common to all cases, extending the reasoning beyond the range in which it originated, obtaining broader results than the particular cases and providing a direct expression that allows generating any term (Radford, 2018). These characteristics allow differentiating strata of generalisation: *pre-algebraic*, when there is no deduction or analytical treatment; *proto-algebraic*, when what is deduced is expressed through particular instances of the variable and incipient analytical features are observed; *algebraic*, when the general rule is used analytically to deduce canonical forms.
- *Structural reasoning*. An algebraic structure is a systematic entity determined by mathematical objects, their binary relationships, the operations on those objects, the properties of those relationships and operations, and the correlations among them. In this way, structural reasoning, understood as reasoning with and about these structures, admits different degrees, according to the generality of the objects involved and the local or global nature of the use of properties and relationships.
- *Functional reasoning*. Functional reasoning involves the construction, description, and reasoning with functions and their elements (variable, domain, image, correspondence rule). It involves expressing relationships between quantities that vary together and using such expressions to analyse the behaviour of a function. Burgos et al. (2024) establish strata of functional reasoning based on: 1) degree of intensity(recognition of the relationship as a set of particular instances or as a general rule) and contextual or non-contextual nature of the rule, 2) type of functional relationship (recursive pattern, covariation, correspondence), and 3) analytical treatment involved in its possible transformation.

- *Analyticity*. *Analytical reasoning* is conceived in terms of the operative character of indeterminate objects through the application of properties of operations. It supports the transformations and equivalence aspects of equations and inequations, and their resolution, although not all equations require the same treatment. Equations of type $Ax + B = C$ (arithmetic equations) can be solved by inverting the operations, while those of type $Ax + B = Cx + D$ (algebraic equations) require a treatment of the unknown. In addition, while an equation can be solved by treating the literal symbol as an unknown, in an inequality, the letter serves as a variable, which entails greater analytical complexity (Burgos et al., 2024).

Table 1

Description of the extended EAR levels model (Burgos et al., 2024)

Level	Essential features
	Pre-algebraic activity
0.I	• *Arithmetic generalisation*. A relationship is detected and applied locally, only in some cases. • *Particular recursive reasoning*. A recursive pattern is conceptualised as a sequence of particular cases, without recognising the regularity that defines the pattern as a generality in unitary form. • Arithmetic operations with grade 1 intensives (natural numbers).
0.II	• *Sophisticated arithmetic generalisation*. A structural view of relationships is recognised, allowing the common characteristic (generality) to be used to determine any term, although generality is not explicitly expressed. • *Particular functional reasoning*. The functional relationship is concretised as a set of particular relationships between the corresponding specific values. • *Pseudo-structural reasoning*. Numerical relationships are intended to express generality, and expressions with numbers are analysed by their structure, not only to operate with them.
	Primary proto-algebraic activity
1.I	• *Factual generalisation*. The generality is determined by the operational scheme, the description of procedures or actions that may be carried out repeatedly, without the indeterminate being incorporated into the discourse. • *Incipient structural reasoning with grade 1 intensives*. Properties of operations and relationships with classes of natural numbers are involved and applied locally, without correlation between the two.

1.II	• *General recursive reasoning*. A recursive pattern is made explicit as a general rule between successive arbitrary values without reference to particular cases. • *Structural reasoning with grade 1 intensives*. Properties of operations and relationships with classes of natural numbers are generally applicable and can be correlated.
Advanced proto-algebraic activity	
2.I	• *Contextual generalisation*. A general rule is deduced and expressed, naming the indeterminate by locative and generic terms, although linked to the context. • *Incipient functional reasoning*. The general relationship is recognised qualitatively or through a set of cases, but without expressing the transformation between the specific quantities. • *Resolution of equations of type* $Ax + B = C$.
2.II	• *Symbolic-contextual generalisation*. The general rule is deduced and expressed in symbolic language, but remains linked to the context, without recognising its canonical form. • *Partial functional reasoning*. The functional relationship is identified, and the transformation between specific quantities is expressed, without articulating the domain, image, and correspondence rule. • *Resolution of inequalities of type* $Ax + B < C$.
Algebraic activity	
3.I	• *Symbolic generalisation*. The general rule is recognised as a new emerging unitary entity that is symbolically materialised and supports transformations into equivalent or canonical forms of its representation. • *Complete functional reasoning*. The general correspondence relationship among arbitrary quantities is recognised and made explicit symbolically. Domain, image, and correspondence rule are perfectly determined, and it is possible to transform the rule to obtain equivalent expressions. • *Resolution of equations of type* $Ax + B = Cx + D$.
3.II	• *Systemic functional reasoning*. The function intervenes as an object, with its own properties and multiple representations. Properties of operations and relationships of functions are recognised and applied locally. • *Resolution of equations of type* $Ax + B < Cx + D$.

Table 1 shows the main characteristics of each sublevel of this model. The interested reader can find in the appendix examples of student responses to one of the problems proposed in the research and the assignment of EAR levels.

METHODOLOGY

Since the purpose of the research is to evaluate the algebraic reasoning (knowledge and levels of EAR) that students in the first three years of CSE mobilise when solving different algebraic problems, we adopted a methodological approach that combines qualitative and quantitative techniques (Leavy, 2022).

235 CSE students from public schools in the provinces of Almería, Granada, and Jaén participated. The centres and schools were selected based on their availability and tutors' interest in participating in the research. We ensured student participation from each grade at each centre, resulting in 9 groups: 91 students in 1st (aged 12–13), 47 in 2nd (aged 13–14), and 97 in 3rd (aged 14–15). The difference in the size of the sample in the 2nd versus the 1st and 3rd grades is due both to the fact that the groups were less numerous (18 in one and 23 in the other two) and that in one of them, and for reasons unrelated to the research, only the data of six students were collected.

Data collection was conducted during a 60-minute session in the last weeks of the first quarter. Participation was voluntary, and the confidentiality and anonymity of the data collected were guaranteed at all times. Students were informed of the importance of their interest in the resolution and that their answers would not affect their scores. When the data were collected, the 1st-graders had not received any algebra training. The 2nd graders' prior knowledge was narrowed to what they had acquired in 1st grade regarding algebraic expressions and first degree equations. Likewise, the 3rd graders' knowledge came from the levels they had previously attended: first and second degree equations, the concept of function, dependent and independent variables, algebraic expressions, and the representation of functions (among others).

Data collection instrument

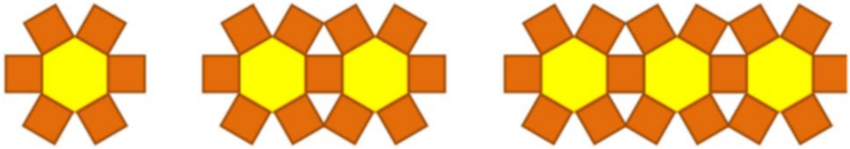
For data collection, we prepared the set of problems, as shown in Figure 1. The instrument is designed to activate different aspects of the EAR, taking into account the essential features of generalisation, functional reasoning, and analyticity established in the theoretical framework.

Figure 1

Data collection instrument

Problem 1. The daisy chain

Julio made a daisy chain with the blocks with the following pattern:



1 flower 2 flowers 3 flowers

a) How many hexagons and squares will Julio need to make a daisy chain with 7 flowers? Explain your answer.
b) How many hexagons and squares will Julio need to make a daisy chain with 20 flowers? Explain your answer.
c) For any number of daisies they give you, how do you calculate the total number of blocks Julio will need for his chain? **Explain** your answer.
d) Is it possible to make a daisy chain that uses exactly 100 squares? **Explain** your answer.

Problem 2. Stools and chairs

There are six seats between chairs and stools. Chairs have four legs and stools have three. In total, there are 20 legs. How many chairs and how many stools are there? **Explain** your answer.

Problem 3. The deal

Juan has saved some money (he only has euros, not cents). Her grandmother wants to reward him for helping her with the garden. She offers him two deals:

(a) Deal 1: "I'll double the money you have" or
(b) Deal 2: "I give you three times your money, and you give me 7 euros".

What deal should he choose? **Explain** your answer.

Problem 4. On the way to school

Andrés goes to a school that is 480 m from his home. On the way, he stops to pick up his friends Víctor and Elena. First, he stops to pick up Victor. Then they travel 52 m. Next, they pick up Elena. The path they have left to the school from that point exceeds three times the distance already travelled by 20 m. What is the distance between Elana's house and school? **Explain/show** how you got the distance.

Problem 1, adapted from Wilkie (2016), aims to evaluate functional thinking through growth patterns: a) and b) they seek to extend the growth pattern (near and far generalisation, respectively), c) it seeks for students to derive the general rule that describes the relationship between the number of hexagons and squares of the chain of flowers and their position in the sequence;

d) it delves into the analysis of the pattern through the inverse relationship. Problem 2, taken from Godino et al. (2014), leads, a priori, to a system of linear equations with two unknowns, although solutions with different levels of EAR are expected. Problem 3 is adapted from Brizuela and Earnest (2008) and has been used in prior studies investigating functional thinking in primary school students (Moreno et al., 2016; Pinto and Cañadas, 2021). In this problem, two functions (linear and affine) must be compared, relating the money Juan has (independent variable) to the money he receives (dependent variable) under two different deals. Finally, Problem 4 is extracted from the 5th-grade mathematics textbook of Editorial SM (Morales et al., 2018). The interest of this problem lies in the fact that, although it is possible to expect a modelling by means of an equation of an algebraic nature ($Ax + B = Cx + D$), it is possible for students to employ proto-algebraic strategies, especially an incipient structural reasoning (Burgos et al., 2024) about the relationships given in the problem, as well as the use of diagrams.

Analysis

Two researchers independently conducted a content analysis of half of the students' productions, identifying types of answers and errors for each problem and assigning a degree of correction and a level of EAR to the mathematical practices. The results were then shared and discussed, and the group agreed on the categorisation and valuation. In the event of a lack of consensus (which occurred only in the assignment of the EAR level in some student solutions), there was support from an external researcher, an expert in the subject. After that, the analysis of the remaining answers was completed.

To assess the degree of correctness of the answers, the following *a priori* categories were considered: *correct*, the result is correct, and its justification is adequate; *partially correct*, the result is correct, but does not justify it or does so incompletely; *incorrect*, otherwise.

On the other hand, the EAR levels were assigned to the global resolution of each student for each problem, based on the highest level activated in their response, in accordance with the extended EAR model (Burgos et al., 2024). These levels were assigned to solutions in which a strategy for assessing the algebraic character of the mathematical practice developed was explicit.

After coding the qualitative analysis answers, the data were statistically treated to examine possible trends in the development of algebraic reasoning and the degree of correction by school grade, as well as the possible relationship between these variables. For this, values were assigned to the correction categories: 0 not answered, 1 incorrect, 2 partially correct, 3 correct; as well as to the levels of algebraic reasoning: 0 unanswered (SR), 1 not codable at any level (NoAp), and from value 2, the different levels of algebraic reasoning were coded (from 0.I to 3.II), according to the assumed theoretical model (Table 1). This coding reflects the hierarchical nature of the model. It allows it to be treated as an ordinal variable in statistical analyses, in line with mixed-methods research that combines qualitative coding and quantitative analysis (e.g., Leavy, 2022).

In the analysis, non-parametric tests were used for different and complementary purposes. On the one hand, the Jonckheere-Terpstra (J-T) test is used to assess whether there is a systematic increase in EAR levels or in the degree of correction across school years. On the other hand, the relationship between the EAR level and the degree of correction was analysed using Kendall's Tau-b ordinal correlation coefficient, which quantifies the intensity of the association while accounting for the ordinal scale of the data and the presence of ties.

RESULTS

Degree of correctness in student responses

Table 2 provides an overview of students' success in problem solving.

Table 2

Distribution of answers according to degree of correction and grade

Grade	Correct	Partially correct	Incorrect	No answer	Total
1st CSE	142 (22.29%)	50 (7.85%)	307(48.19%)	138 (21.66%)	637
2nd CSE	76 (23.10%)	21 (6.38%)	105 (31.91%)	127 (38.60%)	329
3rd CSE	243 (35.79%)	55 (8.10%)	295 (43.45%)	86 (12.67%)	679
Total	461	126	707	351	1645

Note: Percentage of total responses per grade.

Only 28.02% of the answers analysed were correct. The highest percentage of erroneous answers was observed in the 1st CSE, although the 2nd CSE had the highest blank response rate. In 3rd CSE, there is a higher proportion of correct answers and a considerable decrease in the number of unanswered problems. However, this greater willingness to face the problems did not guarantee success, as evidenced by an incorrect-answer rate of 43.45%.

Table 3

Distribution (in percentage) of the degree of correction according to problem and grade

		1st CSE (n=91)	2nd CSE (n=47)	3rd CSE (n=97)	Total (n=235)
P1a	C	28.6	38.3	45.4	37.4
	PC	2.2	0.0	2.1	1.7
	I	59.3	36.2	52.6	51.9
	SR	9.9	25.5	0.0	8.9
P1b	C	22.0	27.7	37.1	29.4
	PC	2.2	0.0	0.0	0.9
	I	60.4	40.4	57.7	55.3
	SR	15.4	31.9	5.2	14.5
P1c	C	7.7	12.8	24.7	15.7
	PC	3.3	2.1	7.2	4.7
	I	62.6	29.8	49.5	50.6
	SR	26.4	55.3	18.6	28.9
P1d	C	12.1	14.9	26.8	18.7
	PC	29.7	12.8	28.9	26.0
	I	22.0	27.7	30.9	26.8
	SR	36.3	44.7	13.4	28.5
P2	C	63.7	55.3	72.2	65.5
	PC	2.2	6.4	2.1	3.0
	I	30.8	12.8	15.5	20.9
	SR	3.3	25.5	10.3	10.6
P3	C	20.9	10.6	30.9	23.0
	PC	15.4	14.9	14.4	14.9
	I	56.0	40.4	49.5	50.2
	SR	7.7	34.0	5.2	11.9
P4	C	1.1	2.1	13.4	6.4
	PC	0.0	8.5	2.1	2.6
	I	46.2	36.2	48.5	45.1
	SR	52.7	53.2	36.1	46.0

Note: C, correct answers; PC, partially correct answers; I, incorrect answers; SR, no answer.

The information in Table 3 shows a relationship between the school level and the degree of correction achieved, although it is neither linear nor consistent across all problems. Specifically, although globally more than half of students answered the first three sections of Problem 1 (P1a: close generalisation, P1b: distant generalisation, P1c: obtaining the general rule) incorrectly, the results were better in the 2nd CSE than in the 1st and 3rd CSEs.

In most cases, the students believed that to obtain the number of squares needed for a given number of flowers, that number must be multiplied by six, without identifying the shared squares in the pattern. In the last section (P1d), the percentage of incorrect answers decreased in favour of partially correct ones. In this case, they based it on the fact that, according to section b, 101 squares were necessary for the 20 flowers, so a chain with 100 squares would not be possible.

Problem 2 (stools and chairs) had the highest rate of correct answers, both globally (65.5%) and in each grade, and the lowest rate of incorrect answers (both globally and in 2nd and 3rd grades). The percentage of blank answers in this problem was also among the lowest. In the erroneous answers, and similarly across the different grades, students often determined the number of chairs and stools to be 20 legs without taking into account the total number of seats, or divided 20 by 4 and 3, assuming the quotients gave the numbers of chairs and stools, respectively. Unlike the results in Problem 1, in this case, the grade with the lowest percentage of correct answers was 2nd CSE, which also had the lowest percentage of incorrect answers and the highest percentage of unanswered items (a pattern reiterated across several of the items analysed).

Similarly, half of the students answered Problem 3 (the deal) incorrectly, with a lower percentage of incorrect answers (though higher than the unanswered rate) in 2nd grade than in the other two. The students opted for one of the deals (usually deal 1) based on one or more specific values of the independent variable, without recognising that the amount received under each deal depended on the initial amount of money, or describing this functional relationship.

The results from Problem 4 (on the way to school) were especially alarming, with 91.1% of students either not answering or answering incorrectly. Students showed difficulty interpreting the relationships between the distances established in the problem and recognising the unknown. For example, they considered that 480 meters was the distance between Andrés's and Víctor's houses, incorrectly interpreted "exceeds by 20m", or assumed that the tripled distance was 52m.

Level of algebraic reasoning involved in resolutions

Table 4 shows the frequency of practices among students at a given level of EAR, by grade.

Table 4

Distribution of answers according to degree of correction and grade

Grade	n*	Levels of EAR							
		0.I	0.II	1.I	1.II	2.I	2.II	3.I	NoAp**
1st CSE	300	47%	11%	10%	1%	16%	5%	0%	10%
2nd CSE	122	39%	14%	9%	7%	11%	6%	0%	15%
3rd CSE	339	35%	5%	12%	4%	17%	11%	10%	5%
Total	761	41%	9%	11%	3%	16%	8%	4%	9%

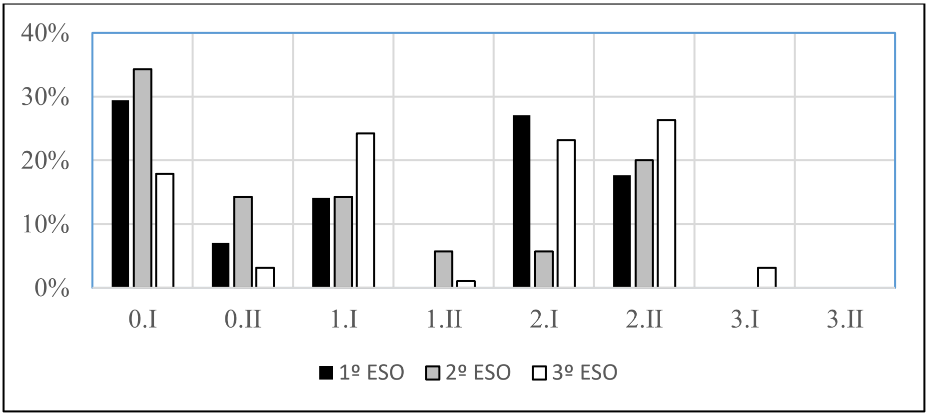
Note: *Number of students per school grade who have responded to any section of problems 1, 2, 3, or 4. **Responses without EAR assigned for not including an explicit strategy.

It can be seen that half of the students' mathematical practices were pre-algebraic (levels 0.I and 0.II), with a lower prevalence in third grade than in previous grades. Practices of a primary proto-algebraic nature (levels 1.I and 1.II) are inferior in all grades to advanced proto-algebraic (levels 2.I and 2.II) and algebraic practices were minimal (5% of the total), only recorded in 3rd CSE (10% of the practices in this grade).

As shown in Figure 2, a considerable percentage of the practices developed by the students to answer Problem 1 corresponded to proto-algebraic levels (59% in 1st CSE, 46% in 2nd CSE, 74% in 3rd CSE), with advanced levels being higher in the three grades (2.I and 2.II). In the practices of level 1.I, students generalised the operational scheme that allows determining the number of squares necessary for a given flower, always linked to the particular numerical value (factual generalisation, according to Table 1). In the practices of level 2.I, they obtained the general rule (correct or incorrect) that enables determining the number of squares, expressed in generic terms but linked to the context (contextual generalisation). In the practices of level 2.II, students deduced the general rule in symbolic language, preserving the spatio-contextual dimension and without operating with the literal symbol to obtain canonical forms of the rule (symbolic-contextual generalisation).

Figure 2

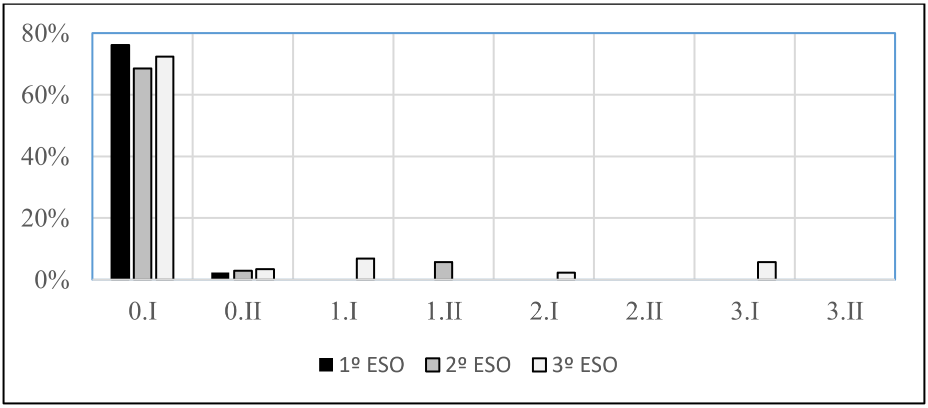
Levels of EAR in Problem 1



The proposed solutions to Problem 2 corresponded essentially to level 0.I (73% of the total), with similar values in all grades (Figure 3). In general, the students solved the problem through trial-and-error strategies, operating with the numbers without demonstrating structural treatment or using relationships or properties among them (as observed in the scarce responses at incipient proto-algebraic levels). Only in the 3rd grade did some students raise and solve the associated system of equations (level 3.I).

Figure 3

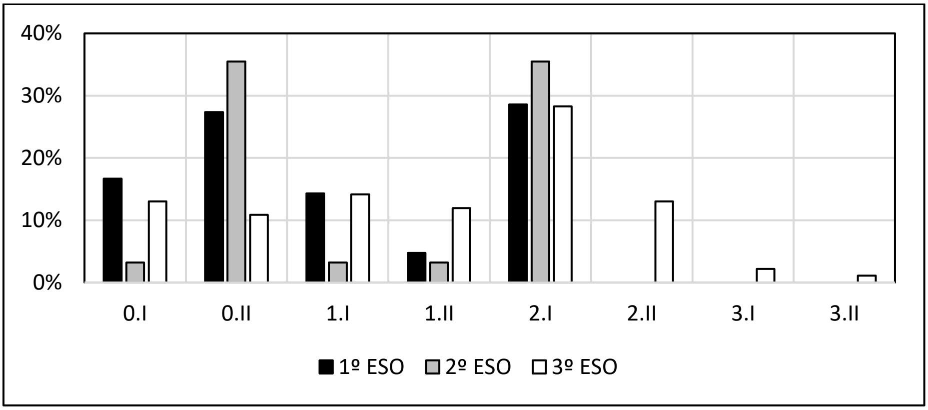
Levels of EAR in Problem 2



In Problem 3 (Figure 4), greater diversity of EAR levels was observed across the practices, along with a progressive decrease in the frequency of pre-algebraic practices and an increase in proto-algebraic practices, especially at level 2.I, as one progresses through the grades.

Figure 4

Levels of EAR in Problem 3

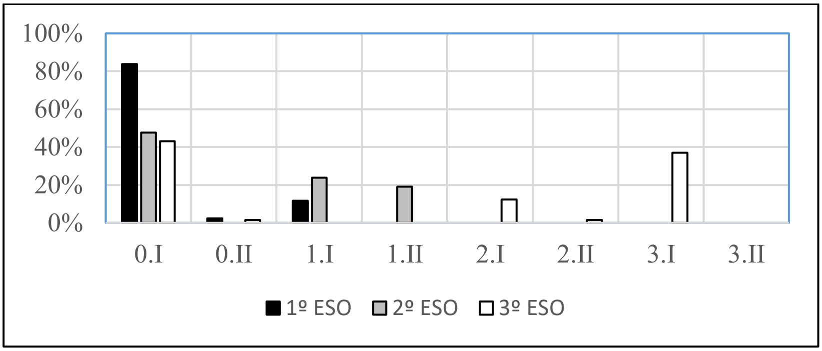


In the practices of level 0.II (21% of the total) students looked at specific values of the variables to determine the best treatment, demonstrating particular functional reasoning (see Table 1). In those of level 2.I (29% of the total), an incipient functional reasoning was observed (Table 1), that is, the students recognised the dependence between the money variables that Juan has and the money he receives, determining from what value treatment 2 is better, although they did not explicitly describe the functional relationships (something that was only observed in the few practices of level 3.I or 3.II, in 3rd CSE).

Finally, Figure 5 shows that most of the practices developed by the students to solve Problem 4 corresponded to a pre-algebraic level (59% of the total), although their frequency decreased as the grades progressed (86% in 1st CSE, 48% in 2nd CSE, 45% in 3rd CSE). In these practices, students used numerical quantities to obtain consistently incorrect results by failing to interpret distances and their relationships properly. In level 1.I practices, they resorted to converting between natural and diagrammatic language to model the situation, but this did not allow them to reach the correct solution. In those practices of level 1.II, students relied on the diagrammatic register to symbolically propose an incomplete algebraic expression (where the unknown is recognised) or an equation that they did not solve. The practices at an advanced proto-algebraic level corresponded to students of 3rd CSE, who incorrectly raised and solved an equation of the type Ax+B=C, and only those who raised and solved an equation of the type Ax+B=Cx+D reached level 3.I.

Figure 5

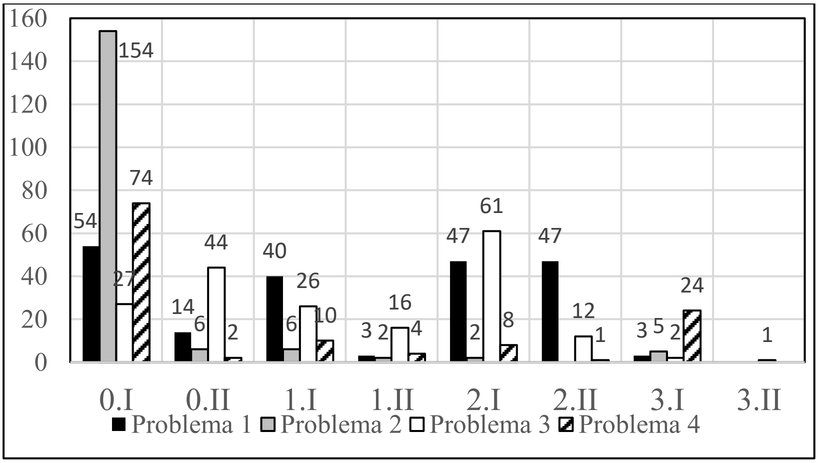
Levels of EAR in Problem 4



In short, Figure 6 shows the frequency of practices at each level of EAR for each problem, highlighting the evident concentration at pre-algebraic levels and a certain presence of practices at the advanced proto-algebraic level in the problems related to functional reasoning (Problem 1 and Problem 3).

Figure 6

Concentration of EAR levels according to the problem



Analysis of the relationship between the degree of correction and school year

To assess the progress in the degree of correction in the resolutions (in general and for each specific problem) relative to the educational grade, the Jonckheer-Terpstra trend test was applied. The overall average degree of correction across all tasks per student was 1.42 (SD = 0.67), indicating predominantly low performance, although with variability between subjects.

Table 5

Jonckheere-Terpstra test considering the mean of the degree of correction

	Mean degree of correction
Number of levels in progress	3
N	235
Observed J-T statistic	11011.00
Average J-T statistic	8831.50
Statistical Standard Dev. J-T	557.47
Dev. J-T statistics	3.91
Sig. asin. (bilateral)	<0.001
a. Grouping variable: Grade	

The Jonckheere-Terpstra test shows bilateral significance (p-value < 0.001 in Table 5). This result indicates a statistically significant upward trend in the degree of correction as the grade progresses.

On the other hand, the results of applying this test to each problem (Table 6) indicate a statistically significant upward trend (p-value = 0.028 < 0.05) only in Problem 4 (equations).

Table 6

Jonckheere-Terpstra trend test results according to the problem

	P1a	P1b	P1c	P1d	P2	P3	P4
Number of levels in progress	3	3	3	3	3	3	3
N	235	235	235	235	235	235	235
Observed J-T statistic	9487.50	9406.50	9322.50	9602.00	8175.50	9035.00	9654.50
Average J-T statistic	8561.50	8561.50	8561.50	8561.50	8561.50	8561.50	8561.50
Statistical Standard Dev. J-T	497.25	495.87	507.95	534.88	465.89	512.26	498.49

Dev. J-T statistics	1.86	1.70	1.50	1.95	-0.83	0.92	2.19
Sig. asin. (bilateral)	0.063	0.088	0.134	0.052	0.407	0.355	0.028
a. Grouping variable: Grade							

This finding suggests that students in higher grades performed better on Problem 4, even though it was the most complex. In sections P1a and P1d (patterns), marginally significant trends were detected (p-values 0.063 and 0.052, respectively), which could indicate incipient evolution in the ability to identify and analyse regularities. However, for P1c, there was no significant improvement in performance across school years; although there was a slight upward trend, it was not clear enough to constitute a firm progression (p-value=0.134). Finally, the P2 (system of equations) and P3 (functions) problems showed no significant differences per grade (p-value = 0.407 for P2, p-value = 0.355 for P3).

Analysis of the relationship between the EAR level and the school year

Firstly, the average EAR per student was calculated as the weighted average of the reasoning levels activated across the set of tasks. The results of the Jonckheere-Terpstra test confirmed a significant increasing trend in the average EAR level per student (J-T = 11.705, p-value < 0.001), supporting the hypothesis of a gradual increase in the algebraic reasoning involved in the practices across the grades.

Next, to assess the presence of an ordinal association between the EAR level used in each problem and the school year, Kendall's Tau-b coefficient was computed. In all cases, the results revealed a positive, statistically significant correlation, indicating that as the grade increases, the EAR level involved in problem solving tends to increase. This association was weaker in Problem 2 (Tau-b=0.119; p-value =0.028), moderate in Problem 1 (Tau-b= 0.150, p-value < 0.001) and Problem 3 (Tau-b= 0.162, p-value < 0.001), while Problem 4 showed the strongest association (Tau-b= 0.247, p-value < 0.001), the latter being the one that showed the strongest association.

Analysis of the relationship between the EAR level and the degree of correction

Table 7 shows Kendall's Tau-b coefficients between the EAR level and the degree of correction of the answer for each problem, broken down by grade (1st, 2nd, and 3rd CSE).

Table 7

Correlation between the EAR levels and the degree of correction by problem and grade (Kendall's Tau-b)

Grade	P1	P2	P3	P4
1st CSE	0.490^{**}	0.077^{++}	0.761^{**}	0.947^{**}
2nd CSE	0.673^{**}	0.485^{**}	0.904^{**}	0.833^{**}
3rd CSE	0.446^{**}	0.290^{**}	0.573^{**}	0.841^{**}

Note: **The correlation is significant at the 0.01 level (two-sided). $^{++}$p-value associated with 0.443.

The analysis showed a positive and significant association in virtually all cases. Specifically, the results were particularly consistent in Problems 3 and 4, where the correlations were high across all grades, indicating that, in these tasks, there was a clear correspondence between the EAR level used and the degree of correction achieved. In Problem 1, the association was moderate across all grades and of similar magnitude. In contrast, in Problem 2, the relationship was weaker, particularly in the 1st CSE, where statistical significance was not reached, indicating that this task was solved correctly using pre-algebraic strategies.

FINAL REFLECTIONS

This paper aimed to evaluate the algebraic reasoning employed by students in the first three years of secondary education as they solved problems in different aspects of school algebra: patterns, functions, equations, and systems of equations. We were interested not only in characterising the emerging EAR level in student practices, but also in exploring the relationships among EAR level, the degree of correction achieved, and the school year, and assessing the influence of the type of problem.

The EAR model adopted (Burgos et al., 2024) addresses the degree of generalisation, functional reasoning, structural reasoning, and analyticity of mathematical practices, regardless of the task that motivates them. However, it is undeniable that a more complex task, understood in terms of the objects and processes involved in solving it, can lead to more efficient solutions and higher-level EAR practices.

In general, poor performance was observed in solving the proposed problems, consistent with previous research showing students' difficulties with algebra (Akkan, 2013; Hewitt, 2012). Although the degree of correction in the answers increased with the school year, this increase was not uniform across all problems; a significant trend in the degree of correction with grade was observed only for Problem 4. In the remaining problems, the differences between grades do not reach statistical significance, suggesting that some problems do not clearly benefit from school progress in terms of correction.

Although more than half of the practices identified were arithmetic, the results show a significant evolution in the level of EAR used across the grades, both in the average value of the activated levels and in the frequency distribution of the different levels involved. On the other hand, although the analysis showed a positive and mostly significant association between the EAR level involved in the practices and the degree of correction achieved, the strength of this association varies by problem type and grade. Thus, high or very high correlations across all grades in Problems 3 (functions) and 4 (equations) indicate that success in solving these problems is strongly linked to the use of a higher EAR level. In contrast, in Problem 1 (patterns), moderate and stable correlations in all three grades suggest that a higher EAR favours correction, but is not strictly necessary to obtain correct answers. Overall, the strength of the association between EAR and correction tends to increase per grade.

Specifically, in Problem 1, students had difficulty detecting regularity, so they could not determine the general correspondence rule; the absence of generalisation was a priority pre-algebraic activity in all three grades. In addition, even when they gave a general rule for the pattern, putting into play an advanced proto-algebraic activity (levels 2.I, 2.II) this was not always correct. In addition, no significant improvement in performance was observed with the academic year. This result is similar to those found in research such as Akkan's (2013) and Utami et al.'s (2023) and slightly worse than those obtained by Wilkie (2016).

In Problem 3, the students recognised the functional relationships only through the correspondence between particular values of the variables (pre-algebraic functional reasoning), without conceiving the functions in a general way or comparing them. The correlation between the degree of correction and the EAR level used was strong in the first two years and moderate in the third. Thus, most students opted for one of the treatments without noticing the dependence among the variables, a fact observed by Brizuela and Earnest (2008) when proposing this problem to primary school students. In this case, no significant difference in performance was found among the grades.

The highest rate of correct answers, both globally and in each grade, was observed in Problem 2, where arithmetic resolutions predominated, a result consistent with previous research (Johanning, 2004; Zwanch, 2022). In this case, no significant association was found between the EAR level and the school year or the degree of correction, which shows that students can approach solving systems of equations in a pre-algebraic way, using arithmetic properties to establish relationships among unknown quantities. On the contrary, Problem 4, which involved modelling a situation using equations, was the most difficult, with a very high percentage of incorrect or blank answers. In this case, the high correlation between the degree of success and the level of EAR and the academic year was evident, with correct algebraic solutions in 3rd CSE and incorrect solutions in 1st and 2nd CSE, mostly of an arithmetic nature. A key difference between the two problems lies in their formulation: while the statement of Problem 2 is simple and the unknown values and arithmetic operations involved are clearly identified, the formulation of Problem 4 is complex, making it difficult to recognise the relationships between the unknown quantities.

We are aware that the results depend on the types of problems used in the research and that it is necessary to expand the instrument with new problems that correspond to other approaches to algebra in secondary school (generalised arithmetic, structural thinking, or demonstration) and other contexts of application (geometric, probabilistic). In any case, the results related to the algebraic reasoning of the group of secondary school students deserve some reflection.

Recognising regularities and patterns, and identifying and expressing relationships among variables, are essential aspects of the development of the algebraic sense in schoolchildren (MEFP, 2022a, 2022b). However, the results of Problems 1 and 3 show the students' lack of familiarity with the patterns and suggest the need to reinforce their conceptual understanding of the functions and their components. Specifically, paying attention to the notions of covariation and correspondence, and to the use of multiple representations (diagrammatic, tabular, graphical), could facilitate the transition from particular types of reasoning to more sophisticated levels of functional reasoning involved in solving these problems.

On the other hand, the interpretation and modelling of mathematical or real everyday life situations with symbolic expressions are fundamental characteristics of the algebraic sense (MEFP, 2022b). In secondary education, this practice often leads to the approach and resolution of equations and systems of linear equations. The results of Problem 2 and Problem 4 lead us to recognise the possibility of strategies such as estimation, trial, and error as first approximations, but also the need to reinforce the flexible use of proto-algebraic representations, such as diagrams (used by some students of our study), tabular or graphical, can help students propose and solve equations, in which the symbols refer to the real world and the operations between them make sense.

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AUTHORSHIP CONTRIBUTION STATEMENTS

MB and NT-E conceived and proposed the research (conceptualisation in accordance with the CRediT taxonomy). MB, NT-E and MML-M adapted the methodology and analysed the data obtained during implementation (methodology and formal analysis). MB and MML-M wrote the original version of the article (writing – original draft), and the three authors participated in the review and editing of the final version (writing – review & editing). All authors actively discussed the results and reviewed and approved the final version of the paper.

DATA AVAILABILITY DECLARATION

The data supporting the results of this study will be provided by the corresponding author, NT-E, upon reasonable request.

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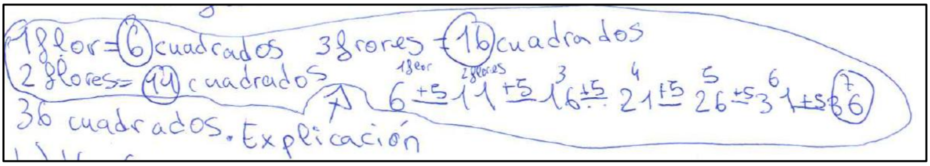
APPENDIX

Mathematical practices of different EAR levels are shown through the students' answers to Problem 1 *The daisy chain*.

- Answer of level 0.I of EAR.

Figure 7

Resolution of E1.1 (1st CSE) Problem 1 (a). Level 0.I of EAR

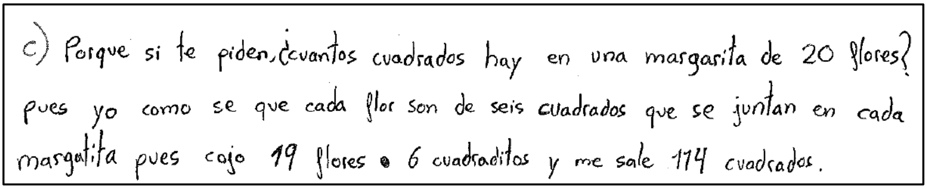

1 flor = 6 cuadrados 3 flores = 16 cuadrados
2 flores = 11 cuadrados 1 flor 2 flores 3 4 5 6 7
6 +5 11 +5 16 +5 21 +5 26 +5 31 +5 36
36 cuadrados. Explicación

To solve section (a) of Problem 1, E1.1. detects a recursive pattern that applies in particular cases, not explaining the generality in a unitary way (particular recursive reasoning).

- Answer of level 0.II of EAR.

Figure 8

Resolution of E1.2 (1st CSE) Problem 1 (b). Level 0.II of EAR

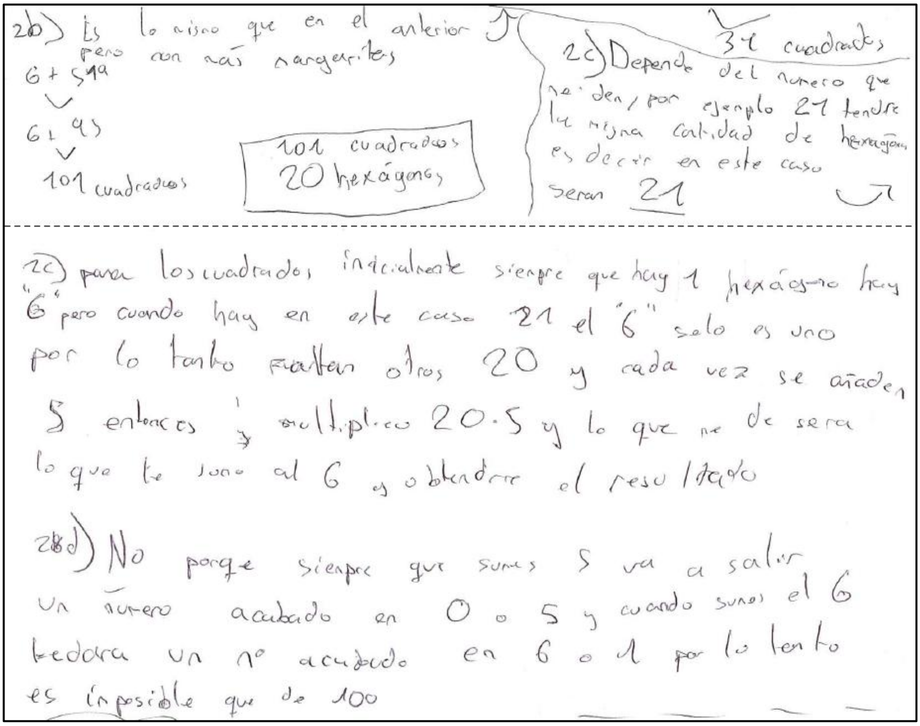

c) Porque si te piden, ¿cuantos cuadrados hay en una margarita de 20 flores? pues yo como se que cada flor son de seis cuadrados que se juntan en cada margarita pues cojo 19 flores • 6 cuadraditos y me sale 114 cuadrados.

When solving section (c), E1.2. uses an example for its structure to explain the general rule detected, without making the generality explicit (arithmetic-sophisticated generalisation). This relationship is specified in the previous sections for chains of 7 and 20 flowers (particular functional reasoning).

- Answer to level 1.I of EAR.

Figure 9

Resolution of E1.3. (1st CSE) of Problem 1(c). Level 1.I of EAR

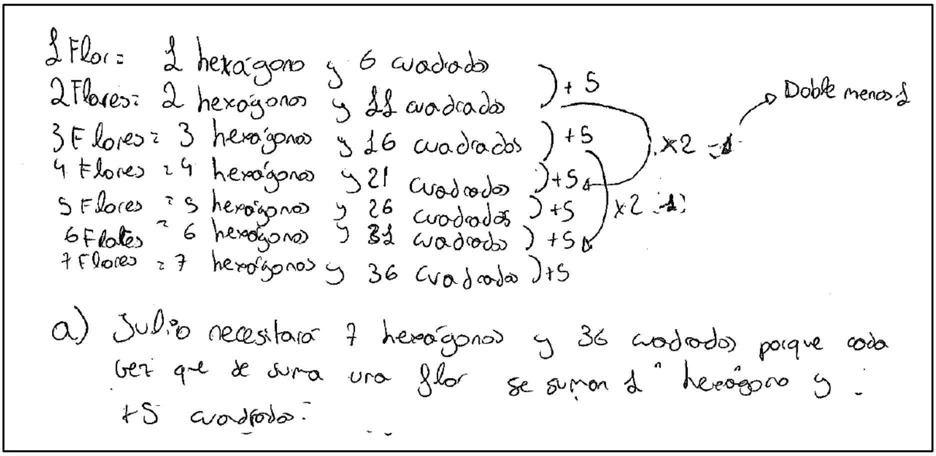

2b) Es lo mismo que en el anterior pero con más margaritas
6 + 5·19
6 + 95
101 cuadrados
101 cuadrados
20 hexágonos
34 cuadrados
2c) Depende del número que nos den, por ejemplo 21 tendrá la misma cantidad de hexágonos es decir en este caso serán 21
2c) para los cuadrados inicialmente siempre que hay 1 hexágono hay "6" pero cuando hay en este caso 21 el "6" solo es uno por lo tanto faltan otros 20 y cada vez se añaden 5 entonces multiplico 20·5 y lo que me de será lo que le sumo al 6 y obtendré el resultado
2d) No porque siempre que sumes 5 va a salir un número acabado en 0 o 5 y cuando sumas el 6 quedará un nº acabado en 6 o 1 por lo tanto es imposible que dé 100

In section (c), E1.3 describes the operational scheme for determining the number of squares from the number of flowers in the chain. Initially, the student indicates that the number of squares “depends on the number they give us”, and exemplifies it in the case of a chain of 21 flowers (in previous sections, he had already done it for chains of 7 and 20 flowers), which implies a factual generalisation of the rule for forming the pattern. In addition, the student relies on properties of operations with natural numbers locally (“whenever you add 5, a number ending in 0 or 5 will come out”) to respond to section (d), which implies an incipient structural reasoning with grade 1 intensives.

- Answer to level 1.II of EAR.

Figure 10

Resolution of E3.1. (3rd CSE) of sections (a) and (b) of Problem 1. Level 1.II of EAR



1 Flor: 1 hexágono y 6 cuadrados) +5
2 Flores: 2 hexágonos y 11 cuadrados
3 Flores: 3 hexágonos y 16 cuadrados) +5
4 Flores: 4 hexágonos y 21 cuadrados) +5
5 Flores: 5 hexágonos y 26 cuadrados) +5
6 Flores: 6 hexágonos y 31 cuadrados) +5
7 Flores: 7 hexágonos y 36 cuadrados) +5

x2 -1 → Doble menos 1
x2 -1

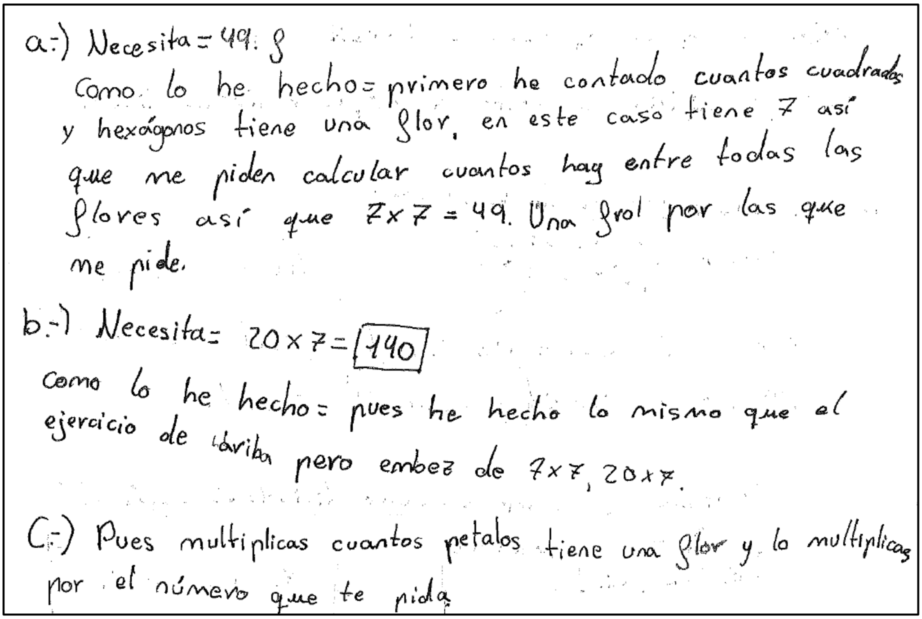
a) Julio necesitará 7 hexágonos y 36 cuadrados porque cada vez que se suma una flor se suman 1 hexágono y +5 cuadrados.

E3.1. detects and makes explicit a recursive pattern that allows it to generate the next term of the series known as the previous value ("every time a flower is added, 1 hexagon and +5 squares are added").

- Answer to level 2.I of EAR.

Figure 11

Resolution of E1.3. (1st CSE) of Problem 1. Level 2.I of EAR


a.-) Necesita = 49. ƒ
Como lo he hecho = primero he contado cuantos cuadrados y hexágonos tiene una flor, en este caso tiene 7 así que me piden calcular cuantos hay entre todas las flores así que 7x7 = 49. Una frol por las que me pide.

b.-) Necesita = 20 x 7 = $\boxed{140}$
Como lo he hecho = pues he hecho lo mismo que el ejercicio de arriba pero embez de 7x7, 20x7.

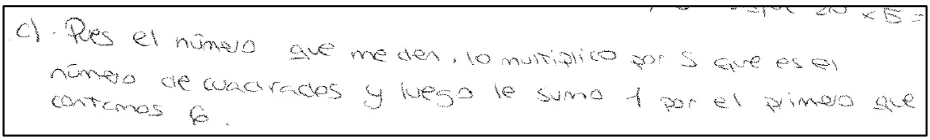
C.-) Pues multiplicas cuantos petalos tiene una flor y lo multiplicas por el número que te pida.

E1.4 expresses in (c) the general rule of pattern formation qualitatively by naming the indeterminate with generic terms ("the number that asks you") but without giving an expression of the transformation with specific quantities ("how many petals a flower has"). It is therefore a contextual generalisation of the formation rule, which involves incipient functional reasoning.

- Answer to level 2.II of EAR

Figure 12

Resolution of E3.2 (3rd CSE) to Problem1 (c). Level 2.II of EAR

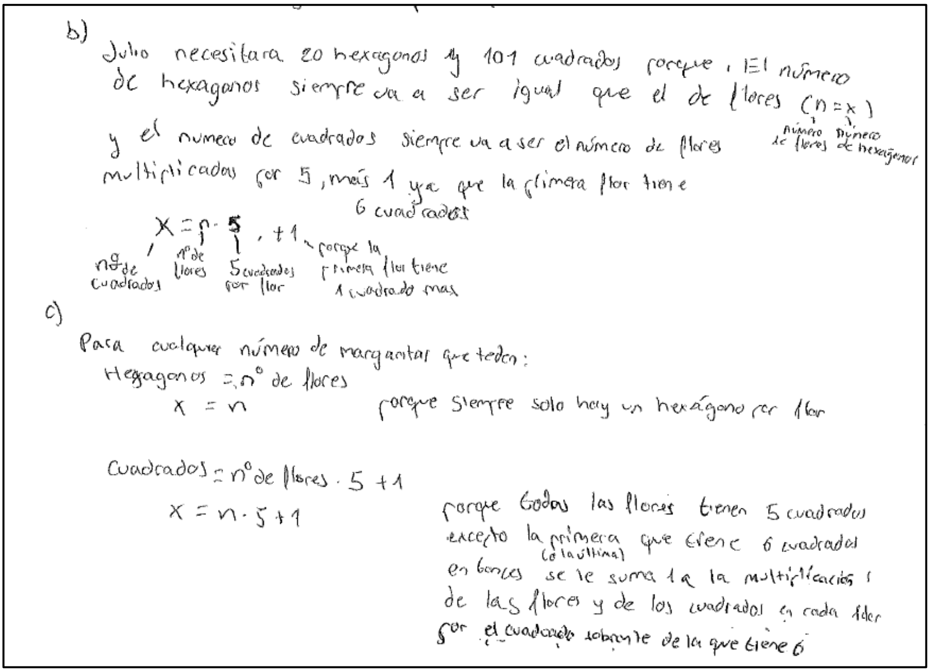

c) Pues el número que me den, lo multiplico por 5 que es el número de cuadrados y luego le sumo 1 por el primero que contamos 6.

E3.2 describes the general rule of pattern formation by using generic terms to name the indeterminate ("the number they give me") and specific quantities for the transformation ("multiply by 5", "sum 1"), but without articulating it with the image (number of squares). It is, therefore, a partial functional reasoning.

- Answer to level 3.I of EAR.

Figure 13

Resolution of E3.3. (3rd CSE) of Problem 1. Level 3.I of EAR


b)
Julio necesitara 20 hexagonos y 101 cuadrados porque, El número de hexagonos siempre va a ser igual que el de flores (n = x)
número de flores
número de hexagonos
y el numero de cuadrados siempre va a ser el número de flores multiplicadas por 5, más 1 ya que la primera flor tiene 6 cuadrados
X = n · 5 · +1
nº de cuadrados
nº de flores
5 cuadrados por flor
porque la primera flor tiene 1 cuadrado mas
c)
Para cualquier número de margaritas que teden:
Hexagonos = nº de flores
x = n
porque siempre solo hay un hexágono por flor
Cuadrados = nº de flores · 5 + 1
x = n · 5 + 1
porque todas las flores tienen 5 cuadrados excepto la primera (o la ultima) que tiene 6 cuadrados entonces se le suma 1 a la multiplicación de las flores y de los cuadrados en cada flor por el cuadrado sobrante de la que tiene 6

E3.3 infers and describes the transformation rule that relates the number of flowers in the chain and the number of squares (hexagons) necessary to build it, expressing it in symbolic-literal notation. It is therefore a complete functional reasoning.