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Contexts of Divisibility as an alternative to the development of Elementary Algebraic Reasoning

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ABSTRACT

Background: Nowadays, the importance of developing elementary algebraic reasoning in the mathematical education of students and teachers is indisputable. It is necessary to identify what other elementary arithmetic content can contribute to the development of elementary algebraic reasoning (EAR) and what the relationship is between the mathematical practices that involve them and the levels of algebraic reasoning. **Objective:** This paper presents a study on the potential of divisibility contexts as a means of promoting elementary algebraic reasoning. **Design:** This study was conducted using a qualitative, theoretical-analytical, and documentary approach. **Analysis**: Using theoretical and methodological notions from the Onto-Semiotic Approach to Mathematical Knowledge and Instruction and, in particular, the proposal for elementary algebraic reasoning levels developed within this theoretical approach, a systematic review and a didactic analysis of research on divisibility was carried out to organize and analyze the situations that are usually proposed in basic education. **Results:** Based on that, partial meanings of divisibility are identified, which in turn can be associated with mathematical practices with features of different levels of algebraic reasoning. It is evident that divisibility, traditionally approached as arithmetic content, can promote processes that are relevant to elementary algebraic reasoning, such as generalization, argumentation, and algebraic symbolization. **Conclusions:** Divisibility is a useful context as an alternative to pattern contexts for the development of elementary algebraic reasoning, confirming that algebraic reasoning can be developed from various elementary arithmetic contents.

Keywords: Divisibility; Elementary algebraic reasoning; Levels of algebraization; Partial meaning, Reference meaning.

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Contextos de divisibilidade como uma alternativa para o desenvolvimento do Raciocínio Algébrico Elementar

RESUMO

Contexto: Atualmente, a importância do desenvolvimento do raciocínio algébrico elementar na educação matemática de alunos e professores é indiscutível. É necessário identificar quais outros conteúdos de aritmética elementar podem contribuir para o desenvolvimento do raciocínio algébrico elementar (RAE) e qual a relação entre as práticas matemáticas que os envolvem e os níveis de raciocínio algébrico. **Objetivo:** Este artigo apresenta um estudo sobre o potencial dos contextos de divisibilidade como meio de promover o raciocínio algébrico elementar. **Metodologia**: Este estudo foi conduzido utilizando uma abordagem qualitativa, teórico-analítica e documental. **Análise**: Utilizando noções teóricas e metodológicas da Abordagem Onto-Semiótica do Conhecimento e do Ensino da Matemática e, em particular, a proposta de níveis de raciocínio algébrico elementar desenvolvida dentro dessa abordagem teórica, foi realizada uma revisão sistemática e uma análise didática da pesquisa sobre divisibilidade para organizar e analisar as situações que são geralmente propostas na educação básica. **Resultados:** Com base nisso, foram identificados significados parciais de divisibilidade, que, por sua vez, podem ser associados a práticas matemáticas com características de diferentes níveis de raciocínio algébrico. É evidente que a divisibilidade, tradicionalmente abordada como conteúdo aritmético, pode promover processos relevantes para o raciocínio algébrico elementar, como generalização, argumentação e simbolização algébrica. **Conclusões:** A divisibilidade é um contexto útil como alternativa aos contextos de padrões para o desenvolvimento do raciocínio algébrico elementar, confirmando que o raciocínio algébrico pode ser desenvolvido a partir de diversos conteúdos aritméticos elementares.

Palavras-chave: Divisibilidade; Raciocínio algébrico elementar; Níveis de algebrização; Significado parcial; Significado de referência.

INTRODUCTION

The definition of algebra and algebraic reasoning in elementary grades has been a topic of interest for the mathematics education community. Consequently, it is widely agreed that algebra is not just a set of rules for manipulating symbols. It is, in essence, a way of thinking and acting that allows mathematical objects, relationships, structures, and situations to be analyzed in a particular way, which promotes teaching based on understanding mathematics (Carpenter, Franke, & Levi, 2003; Blanton & Kaput, 2005; Cai & Knuth, 2011).

Along these lines, Godino and Font (2003) state that algebraic reasoning “involves representing, generalizing, and formalizing patterns and regularities in any aspect of mathematics” (p. 774). On the other hand, Castro et al. (2011) point out that elementary algebraic reasoning is the “system of operational and discursive practices brought into play in the resolution of tasks that can be addressed in basic education in which algebraic objects and processes (symbolization, relation, variables, unknowns, equations, patterns, generalization, modeling, etc.) are involved" (p. 75). This involves working with variables and performing arithmetic operations with them, besides representing situations with expressions and constructing equations. They also refer to the manipulation and simplification of expressions and solving of equations, as well as the interpretation of their results.

In conceptualizing what is meant by algebraic practices in the different proposals, we can find several common elements, one of which refers to the identification of patterns, construction of generalizations, and representation of relationships through symbols (Carraher & Schliemann, 2007). In particular, they include situations that require identifying regularities (e.g., Zazkis & Liljedahl, 2002a; Radford, 2006; Demonty et al., 2018; Bautista-Pérez et al., 2021).

In this context, teachers must have training experiences that enable them to identify tasks that promote the development of algebraic reasoning and investigate the mathematical activity that students must perform to solve them (Pincheira & Alsina, 2021). To this end, proposals have been developed to put this approach in operation (e.g., Najar et al., 2021; Godino et al., 2015; Godino et al. 2014; Castro et al., 2017).

Furthermore, this expanded view of algebra requires adapting problems addressed in other areas (e.g., Burgos, Batanero & Godino, 2022), such as arithmetic, so that they demand the application of algebraic concepts and procedures (Carraher & Schliemann, 2007). Thus, the great potential of arithmetic notions for the development of algebraic reasoning is recognized, since the transition from arithmetic to algebraic reasoning involves a shift from a procedural perspective of operations and relationships to a structural perspective, considering tasks that require making and representing generalizations (Kieran, 1992). In that sense, various extensive studies have been carried out based on situations in arithmetic contexts for the development of algebraic reasoning (e.g., Carraher et al., 2006; Llanes et al., 2022). In particular, it has been proposed to study arithmetic operations from a functional perspective (e.g., Schliemann, Carraher & Brizuela, 2007). Activities involving arithmetic sequences have also been considered, which allow for the exploration of connections between additive and multiplicative structures, as well as various concepts related to number theory (Zazkis & Liljedahl, 2002b).

A systematic analysis of studies on algebraic reasoning developed over the last 10 years reveals that continuous or prior teacher training is necessary to develop algebraic reasoning and that non-routine activities should be used in the classroom (Sibgatullin et al., 2022). Furthermore, Sun et al. (2023) point out that educational practices should include content from elementary arithmetic curricula that promote generalization skills.

This makes it necessary to carry out research on different forms of curricular activity that support the development of elementary algebraic reasoning based on number theory, the operations defined between them, and their properties.

From this perspective, and considering the common elements of different proposals for working on algebraic reasoning in the training of primary and secondary school teachers, the research question that guided this study is as follows:

what other elementary arithmetic contents can contribute to the development of elementary algebraic reasoning (EAR) and what is the relationship between the mathematical practices that involve them and the levels of algebraic reasoning?

This paper presents an exploratory study that aims to investigate the potential of divisibility contexts as a means of promoting EAR. In other words, we seek to determine whether divisibility tasks can be considered algebraic in nature, as long as they allow the manifestation of algebraic characteristics in the mathematical practices carried out during their resolution.

To achieve the stated objective, section 2 presents a review of some research done on divisibility from epistemic, cognitive, and curricular perspectives, highlighting its potential algebraic nature. Section 3 presents the elements of the theoretical and methodological framework considered in this work, which are based on the proposal of levels of Elementary Algebraic Reasoning (EAR) from the Onto-Semiotic Approach to Mathematical Knowledge and Instruction (OSA). Section 4 proposes a reference meaning for divisibility in schools; section 5 associates features of algebraic reasoning with proposed solutions to the different situations that characterize each partial meaning. Finally, the section on conclusions and open questions is presented.

THEORETICAL BACKGROUND

Divisibility is one of the main mathematical concepts in number theory (Dickson, 1919). It is based on the multiplicative structure of integers, prime factorizations (fundamental theorem of arithmetic), and notions of modular arithmetic, such as Euclid's division algorithm (Brown, Thomas & Tolias, 2002).

Working with divisibility situations begins by expressing natural numbers as the product of prime numbers through the operation of multiplication and the use of its properties. The criteria for divisibility by D are justified based on the decomposition of numbers in base 10 and the mod D equivalence relation. Systems of linear Diophantine equations (with integer coefficients and solutions) are solved considering the greatest common divisor and Euclid's algorithm.

These ideas are extended to the ring of integers, Z, but then extended to any ring that is an integral domain. Thus, the underlying algebraic structure corresponds to that of an ideal with unique factorization. In particular, the generation of divisibility criteria has been a topic of interest for the mathematical community. Based on research into the origins of divisibility criteria, Ahuja and Bruening (1999) conclude that the concepts underlying all divisibility criteria tests developed since ancient times correspond to notions of modular arithmetic, albeit with different nuances. The justification of the criteria evolves from studying particular cases to systematizing them in formal language, such as the ones carried out by Pascal, and to the construction of the Number Theory (Oller & Meavilla, 2013). Thus, a rich articulation between arithmetic and algebra is expected when studying situations related to divisibility criteria.

Although divisibility is a mathematical object of the arithmetic framework, as it is first defined in natural numbers when extended to the set of integers, it allows the solution of Diophantine equations, which are situations in the algebraic framework that require the use of properties of the set of integers as a domain of integrity. Therefore, divisibility has an intrinsically algebraic character in the sense that it involves structural properties of the numerical system that transcend operations between numbers, which means that it can be used as a possible problem for the construction of algebraic knowledge (Valiero, 2020).

Furthermore, Zazkis and Campbell (1996) point out that the successful development of conceptual understanding in algebra requires a solid foundation in the conceptual understanding of arithmetic and elementary number theory. Elementary number theory provides teachers with valuable opportunities to enrich their understanding of the properties of multiplication and division, as it requires the application of their knowledge of the multiplicative structure of number sets (Brown, et al., 2022).

In this context, they conducted a study with teachers and found that most of them need to perform divisions to determine whether one number is divisible by another, even in cases where prime factorization is a more efficient procedure. This shows that they do not understand divisibility as a property of natural numbers or a relationship between them. They also note that the use of divisibility criteria without understanding their origin can reduce the understanding of divisibility to a search for digit patterns.

This explains why, when teaching divisibility, teachers prioritize situations that can be addressed with a procedural activity, which in turn reinforces a strictly empirical attitude toward mathematics. In particular, this leads to students forgetting the rules of divisibility due to memorization (Teke & Çalışıcı, 2025).

While Noblet (2016) analyzes in-training primary school teachers' understanding of divisibility, they show a deep knowledge of prime factorization; however, half of them do not use that knowledge to determine divisibility by a particular number. On the other hand, Marieli et al. (2019) characterize the knowledge of secondary school teacher trainers in relation to the proof of Euclid's division algorithm theorem. They find that, although they have knowledge of the inherent mathematical structure, they do not have the expected knowledge when carrying out practices that require demonstrations. This occurs despite divisibility being an ideal context for performing "mathematical tests" (Martin & Harel, 1989).

In relation to understanding divisibility, Espinoza & Pochulu (2020) propose levels of understanding that subjects can achieve when solving divisibility tasks. In this proposal, they consider those who only establish propositional semiotic functions to be at the first level of understanding, those who can establish argumentative semiotic functions based on concepts and propositions to be at the second level, and those who establish formal deductive argumentative semiotic functions to be at the third level.

Various papers on mathematics education focus on studying how divisibility teaching is organized.

Gascón (2001) identifies a mathematical organization that contains types of divisibility problems in natural numbers, present in Spanish secondary schools, as well as the techniques used. The types include determining whether a number is a multiple of another, which requires division; a problem that requires finding a number that is a multiple of others, which requires calculating the least common multiple (LCM); a problem that requires finding a common divisor of a set of numbers, which involves calculating the greatest common divisor (GCD); finding all the divisors of a number that are multiples of another given number and, finally, determining unknown digits of a number, knowing some of its divisors, based on the use of divisibility criteria.

The author finds that these types are studied separately and solved using arithmetic techniques, which are considered pre-algebraic, and do not allow for the manipulation of the overall structure of the types of problems. In this context, he develops a proposal to rescue the algebraic character that can be attributed to divisibility. This is done through the progressive expansion and completion of activities, so that the structure inherent in the types of tasks must be symbolically materialized through formal rules, carrying out justifications and even demonstrations.

Meanwhile, from a phenomenological perspective, López, Castro and Cañadas (2016) identify four contexts for the use of divisibility, which are: arithmetic operations (breaking down a number into factors, dividing or multiplying two numbers), the divisor relationship (a divides or is a factor of b), the multiple relationship (b is a multiple or is divisible by a), and the cardinal relationship (breaking down a number into prime factors and determining its number of divisors).

Espinoza (2019) reviews Argentinean school textbooks, curriculum designs, and research papers on divisibility and proposes an epistemic configuration consisting of 14 types of tasks, as well as definitions/concepts, propositions, procedures, arguments, language, and their relationships.

Also, in the Argentinean context, Muñoz et al. (2021) analyze primary and secondary school textbooks and identify four mathematical organizations associated with divisibility in integers, namely: i) searching for multiples and divisors; ii) calculating the greatest common divisor and the least common multiple; iii) use of divisibility criteria; and iv) applications of the division algorithm and the fundamental theorem of arithmetic. However, they point out that the tasks do not allow for exploration of the scope of the techniques used to solve them, associating each task with a single technique, nor is emphasis placed on the interpretation of results, and open modeling situations are not identified.

Llanes, Pino-Fan and Ibarra (2022) identify only three types of problem situations related to divisibility in Chilean school textbooks, which are attributed an arithmetic character. Therefore, it is considered that the mathematical practices developed only show features of algebraic reasoning level 0 or 1 in cases where it is necessary to identify the structures involved.

In general, divisibility is taught within the field of arithmetic, with an emphasis on determining common multiples and divisors, using divisibility criteria, and solving numerical problems related to these concepts, resulting in particular numbers.

On the other hand, Sadovsky and Sessa (2005) point out that teachers find it rewarding to work on divisibility criteria because they are transformed from "rules that are applied" to statements whose validity can be justified through reasoning based on simple knowledge that even a primary school teacher can understand. In the same vein, Cambriglia and Sessa (2011) find that if activities on divisibility and multiplicative decompositions are properly managed, promoting collective constructions in the classroom, regularities can be perceived, giving room to the development of conjectures associated with generalization processes, which characterize algebraic reasoning.

The review of the literature shows that divisibility is a field rich in problem situations with different degrees of conceptual and formal complexity. In particular, situations involving demonstrating or proposing divisibility criteria require processes of generalization and formalization, which are characteristic of algebraic practices. In this transition, divisibility reveals properties that can favor the progressive emergence of forms of algebraic reasoning.

THEORICAL FRAMEWORK

From the Onto-Semiotic Approach to Mathematical Knowledge and Instruction (OSA) (Godino et al., 2024), the referential meaning of a mathematical object is conceived as the system of mathematical practices, objects, and processes that historically give it meaning (Pino-Fan, Godino & Font, 2011; Pino-Fan, 2017). This way of understanding meaning corresponds to a pragmatic view of the mathematical object, considering what can be done with it; thus, it seeks to answer the question: What are the different meanings of a mathematical object in a given institutional context or framework? (Pochulu & Font, 2022).

In particular, in the case of divisibility, this meaning can be organized around various epistemic configurations or types of situations, each of which mobilizes different objects (numbers, operations, relations) and mathematical processes (representation, generalization, argumentation, formalization, etc.).

Based on the various existing theoretical proposals for understanding algebraic reasoning, the OSA has contributed specific characterizations of the objects and processes involved in algebraic activities. As long as we recognize the presence of intensive objects in a mathematical practice, at any of its levels of generality or intension, we are able to attribute a certain degree of algebraization to that practice, whether the intensive is expressed alphanumerically or not (Godino et al., 2012, 498-499).

In this context, they have formulated a proposal for levels of algebraization (Table 1) designed to describe how elementary algebraic reasoning (EAR) develops gradually, initially in primary school (Godino et al., 2012, 2014), and then extends to secondary school and high school (Godino et al., 2015).

Table 1

Levels of Elementary Algebraic Reasoning proposed by OSA

Level	Description of levels
0	It is associated with a mathematical practice when (particular) extensive objects are involved, which are expressed through natural, numerical, iconic, or gestural language.
1	It is associated with (general) intensive objects whose generality is explicitly recognized through natural, numerical, iconic, or gestural language. Symbols that refer to recognized intensives may be involved, but without operating on those objects.
2	It is associated with indeterminate or variable expressions of grade 2 intensives. Symbolic-literal language is used to refer to recognized intensives, although linked to spatial and temporary context information.
3	It is associated with the use of indeterminates, unknowns, equations, variables, and particular functions. Symbolic-literal language is used; symbols are used analytically, without referring to contextual information analytically, without referring to contextual information.
4	When the mathematical activity performed includes the use of parameters as a numerical record or to express families of equations and functions, signs of a higher level of algebraic reasoning are identified.
5	A higher level of semiotic complexity is recognized when the mathematical activity involves (syntactic) analytical calculations involving one or more parameters, with which operations are performed and relationships are established.
6	When the mathematical activity involves algebraic objects and processes of a higher degree of generality, such as the treatment of algebraic structures or function algebra.

Note. Adapted from Godino et al. (2014) and Godino et al. (2015)

The EAR model considers a level 0 of absence of algebraic reasoning, and levels 1 to 6 to describe an evolution based on the type of mathematical objects and processes involved in mathematical activity. The evolution in levels of algebraic reasoning 1, 2, and 3 is associated with the presence of second-degree intensive objects (classes or types of first-degree intensives, such as natural numbers), which are expressed alphanumerically and operated on analytically (syntactically). On the other hand, progression in levels 4, 5, and 6 is associated with the use of parameters and their treatment since their use encourages the reification of formulas and contributes to generalization (Godino et al., 2015).

The different levels of algebraization are directly related to the degree of generality achieved by mathematical concepts. Progress in this area can be measured by the ability to use multiple registers of semiotic representation (different systems to express mathematical ideas) and to perform operations and conversions between them fluently. For Godino et al. (2015), this illustrates the stages in which intensive mathematical objects—those that are more abstract and complex—become firmly established or "reified"; that is, they become more concrete entities.

METHODOLOGY

This study was conducted using a qualitative, theoretical-analytical, and documentary approach (Creswell, 2009; Cohen et al., 2011), focused on the didactic and epistemic analysis of the notion of divisibility in basic education and teacher training contexts. This is not an empirical study with data collected in the classroom, but rather a conceptual and analytical investigation whose purpose is to characterize the potential of certain arithmetic content, particularly divisibility, for the development of elementary algebraic reasoning (EAR). To this end, we draw on the notion of reference meaning, epistemic configurations, and the levels of elementary algebraic reasoning developed in previous works (Godino et al., 2012; 2014; 2015).

The study was structured in four main phases, which were carried out in an iterative and coherent manner.

- Review and systematization of the literature. Analytical review of previous research on divisibility from epistemological, cognitive, didactic, and curricular perspectives. The corpus considered included: research articles on mathematics education, textbook analysis, curricular proposals, and studies on teacher training. In this phase, it was possible to identify recurring types of problem situations associated with divisibility, predominant resolution techniques, emphasis on arithmetic or algebraic characteristics, and relationships with processes of generalization, symbolization, and argumentation. This phase formed the basis for defining the predominant institutional uses of divisibility and its potential articulation with the EAR.

- Identification of partial meanings of divisibility. The criteria for identifying partial meanings were functional and epistemic, according to OSA notions, considering: the type of mathematical relationship involved (operational, relational, structural), the numerical set involved, the degree of generality of the objects mobilized, and the dominant mathematical processes. As a result, five partial meanings of divisibility are proposed, ranging from elementary arithmetic relations to more general algebraic structures.
- Construction of epistemic configurations associated with problem situations. For each partial meaning, prototypical problem situations were selected and analyzed, representative of common practices in basic education and teacher training. The analysis was carried out by constructing epistemic configurations, identifying the following components: types of situations, languages and semiotic registers, definitions and concepts mobilized, properties and propositions involved, resolution procedures, validation arguments, and underlying mathematical processes (generalization, symbolization, argumentation, formalization). This analysis made it possible to explain the semiotic and epistemic complexity of each type of practice associated with divisibility.
- Analysis of the elementary algebraic reasoning features mobilized. As part of this phase, the epistemic configurations obtained were analyzed in light of the model of elementary algebraic reasoning levels proposed by the OSA. The central criterion was the identification of intensive objects, the degree of generality achieved, and the type of semiotic treatment performed. This analysis showed that situations of divisibility, traditionally approached from an arithmetic perspective, can favor the progressive development of elementary algebraic reasoning, including processes of generalization, use of indeterminates, symbolic treatment, and deductive reasoning.

RESULTS: PARTIAL MEANINGS OF DIVISIBILITY

Based on the review done on the background and the theoretical framework adopted, it is understood that divisibility has different meanings. Below is a proposal for partial meanings of divisibility, framed within the training of elementary and secondary education teachers. Situations associated with each of these meanings are described, along with specific examples. To identify the set of objects that are activated when solving them, epistemic configurations are proposed for the typology of problem situations.

Meaning 1: Divisibility as a relationship between natural numbers, a relationship that defines equivalence classes modulo n.

It corresponds to mathematical practices that involve situations such as those described in S.1.1 and S.1.2.

S.1.1 Given two numbers, determine if one of them is divisible by the other (or a divisor of the other). For example, determine if 391 is divisible by 23.

As an example, epistemic configuration is proposed for the described situation in Table 2.

Table 2

Epistemic configuration for situations where it must be determined whether one number is divisible by another

Primary object/processes	**Description**
Situation	Determine if 391 is divisible by 23
Languages	Multiple; divisor, divisible $391 = \overset{\circ}{23}$ $\quad \begin{array}{l\|l} 391 & 23 \\ \hline 0 & 17 \end{array}$ $\quad 23 \mid 391$
Definitions, concepts	d is a divisor of N if and only if there is a natural number x, provided that N=xd N is a multiple of d if and only if there exists a natural number x such that N=xd
Properties	Division algorithm in N Given a and b in N, with $a>b$, there are natural numbers c and r, provided that: a=$bc+r$, with $0 \leq r < b$

Procedures	Divide 391 by 23. If the remainder is 0, then 391 is a multiple of 23; otherwise, 391 is not a multiple of 23.
Arguments	The answer is justified based on the result of the arithmetic division operation
Processes	Particularization and algorithmization

S.1.2 Find a number knowing which equivalence class modulo N it belongs to. For example, consider the following table with 5 columns and infinite rows, and determine which number is in row 23, column 3.

As an example, epistemic configuration is proposed for the described situation in Table 3.

Table 3

Epistemic configuration for find a number knowing which equivalence class modulo N it belongs to

Primary object/processes	Description
Situation	Consider the following table with 5 columns (columns 0-4) and infinite rows (rows 0-…), and determine which number is in row 23, column 3. 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
Languages	Divisor, multiple, remainder $$22 = (5)(4) + 2$$ $$2\ mod\ 5 = 17\ mod\ 5$$ Arithmetic progression.
Definitions, concepts	Division, equivalence relation modulo 5, arithmetic progression.
Properties	Euclidean division theorem: Given integers a and b, there are integers q and r, provided that $a = bq + r$, with $0 \leq r < b$ The column numbers are generated according to a pattern, and the numbers in each cell can be obtained by associating the column and row numbers with the remainder and quotient of dividing by 5, respectively.
Procedures	The numbers in column 3 form an arithmetic progression with ratio 5. That is, they are all multiples of 5 plus 3. In this case, the term in row 23 is $$3 + (23 - 1)(5) = 113$$

Arguments	Inductive justification based on the identification of a pattern of formation in either a column, row, or the entire table.
Processes	Generalization: implicit recognition of a matching rule.

Meaning 2: Divisibility as an operation between natural numbers that allows a number to be represented in equivalent forms as the product of powers of prime factors

It corresponds to mathematical practices that involve situations such as those described in S.2.1, S.2.2, S.2.3 and S.2.4.

S.2.1 Determine all the divisors of a natural number. For example, determine all divisors of 3300 and indicate how many there are.

S.2.2 Find the greatest common divisor of two or more numbers in a contextualized situation. For example, wooden slats measuring 240 cm and 396 cm are to be cut into smaller slats so that they are all the same size. What is the maximum size that these slats must be so that there is no wood left over?

S.2.3 Find the least common multiple of two or more numbers in a contextualized situation. For example, a lighthouse flashes every 240 seconds and another every 396 seconds. If both flashed at 6:00 p.m., how long will it be before they flash again at the same time?

As an example, epistemic configuration is proposed for the third situations in Table 4.

Table 4

Epistemic configuration for finding the least common multiple of two or more numbers in a contextualized situation

Primary object/processes	Description
Situation	A lighthouse flashes every 240 seconds and another every 396 seconds. If both flashed at 6:00 p.m., how long will it be before they flash again at the same time?
Languages	

	$1- \quad 11 \ \vert \ 11$ $1- \quad 1 \ \vert$ $LCM(240; 396) = 7920$ divisor, multiple, etc.
Definitions, concepts	m is the least common multiple of a and b if m is a multiple of a and b and m is a divisor of any other common multiple of a and b. Coprime numbers.
Properties	Divisibility criteria. If d and d' are common divisors of a and b, then dd' is also a divisor of a and b. The LCM of two numbers whose prime factorizations are $N = p_1.p_2 \ldots .p_n.$ and $M = q_1.q_2 \ldots .q_n$ is obtained by multiplying all the common factors and then the non-common factors.
Procedures	Decompose the given numbers into their prime factors. Given 240 and 396, each one is decomposed into its prime factors, select the common factors: 2, 2 and 3. Then, identify the non-common factors and the least common multiple is $2 \times 2 \times 3 \times 2 \times 2 \times 5 \times 3 \times 11 = 7920$. When finding the LCM of more numbers, a similar procedure is followed.
Arguments	The answer is justified based on the decomposition and identification of common and uncommon factors.
Processes	Modeling and particularization.

Meaning 3: Divisibility as a relationship between integers that allows equations to be solved

It corresponds to mathematical practices that involve situations such as those described in S.3.1, S.3.2 and S.3.3.

S.3.1 Situations that are modeled with equations with coefficients and solutions in the set of natural numbers, with a finite number of solutions. For example, is it possible to get 727 soles using only 2-sol and 5-sol coins? If so, how many different ways can this be done?

As an example, epistemic configuration is proposed for the described situation in Table 5.

Table 5

Epistemic configuration for situations that are modeled with equations in natural numbers, with a finite number of solutions

Primary object/processes	Description
Situation	Is it possible to get 727 soles using only 2-sol and 5-sol coins? If so, how many different ways can this be done?
Languages	Equation $$2x + 5y = 727$$
Definitions, concepts	Division by $\mathbb{N}$, divisibility Linear equation with two unknowns, solution.
Properties	Properties of divisibility: The sum of two even numbers or two odd numbers is an even number; the sum of an even number and an odd number is odd. The product of an even number and an integer is even.
Procedures	The solutions can be obtained by simple inspection. The aim is to find the natural numbers x and y, provided that $$2x + 5y = 727$$ The largest value that y can take is 145, $727 = 5(145) + 2$, where $x = 1$. To narrow down the search, divisibility properties can be used. Since $2x$ is even, $5y$ must be odd for the sum to be odd number 727. Since $5y$ must be odd, y must also be odd. Thus, values $y = 145; 143, 141; \ldots\ldots; 1$ are analyzed And values for $x = 1; 6; 11; \ldots.; 361$ are obtained
Arguments	Since the number of solutions is finite, all solutions are found.
Processes	Generalization, reification and algebraic symbolization, but without treatments.

S.3.2 Situations that are modeled with equations with coefficients and solutions in the set of natural numbers, but with infinite solutions. For example, if photographs are counted in groups of 5, there are 3 left over; if they are counted in groups of 9, there are 6 left over. How many photographs could there be?

As an example, epistemic configuration is proposed for the described situation in Table 6.

Table 6

Epistemic configuration for situations that are modeled with equations in natural numbers, with an infinite number of solutions

Primary object/processes	Description
Situation	If photographs are counted in groups of 5, there are 3 left over; if they are counted in groups of 9, there are 6 left over. How many photographs could there be?
Languages	Divisor, multiple, remainder $N = 5x + 3$
Definitions, concepts	Linear equation with two unknowns, that is, an equation like $ax + by = c,$ with $a, b, c \in \mathbb{Z}$ and solutions in $\mathbb{N}$ Division in $\mathbb{N}$, divisibility
Properties	Euclidean division theorem: Given integers a and b, there are integers q and r, provided that $a = bq + r,$ with $0 \leq r < b$. If a Diophantine equation of form $ax + by = c$ has a solution, then it has infinite solutions. Given the solution to a Diophantine equation of form $ax + by = c$ and a solution $(x_0; y_0)$, then $x = x_0 - kb; \;\; y = y_0 + ka$ is also a solution.
Procedures	$N = 5x + 3; \quad N = 9y + 6$ Then, $5x + 3 = 9y + 6$ $5x - 9y = 3$ A particular solution is $x = 6; y = 3$, because $5(6) + 3 = 9(3) + 6$ A general solution must satisfy $5x - 9y = 3$ Subtracting the equations gives: $5(x - 6) = 9(y - 3)$ Then, $x - 6$ is a multiple of 9: there is m, provided that $x - 6 = 9m$. In the same way, $y - 3 = 5n$. Where $x = 6 + 9m, y = 3 + 5n$, but since

	$5(9m) = 9(5n)$, then $m = n$. Therefore, the solutions to the equation are of the form: $x = 6 + 9m, y = 3 + 5m$, with m as an integer. For m =0: $x = 6,\ y = 3$ and $N = 33$ For m =1: $x = 15,\ y = 8$ and $N = 78$ Infinite solutions are obtained in the form $N = 5(6 + 9m) + 3 = 33 + 45m,$, with $m \in \mathbb{N}$.
Arguments	Deductive logical justification based on identifying a particular solution and generating infinite solutions using divisibility properties.
Process	Generalization, reification, algebraic symbolization and treatments.

S.3.3 Situations that are modeled with equations with coefficients and solutions in the set of integers. For example, analyze if the line with equation $2x + 5y = 727$ passes through points in the second quadrant whose coordinates are integers.

As an example, epistemic configuration is proposed for the described situation in Table 7.

Table 7

Epistemic configuration for situations modelled with equations with coefficients and solutions in the set of integers.

Primary object/processes	Description
Situation	Analyze if the line with equation $2x + 5y = 727$ passes through points in the second quadrant whose coordinates are integers.
Languages	Divisor, multiple
Definitions, concepts	Linear Diophantine equation of the form ax +by =c, with a, b, $c \in \mathbb{Z}$ and integer solutions. Division in $\mathbb{Z}$, divisibility
Properties	Proposition: If ab is divisible by d, and a is not divisible by d, then b is divisible by d. Corollary of Euclid's theorem: Given integers a and b and $c = GCD(a; b)$, there are integers x and y, provided that $ax + by = c$.
Procedures	In this case, we are looking for integers x and y,

provided that

$$2x + 5y = 727$$

Since $GCD(2; 5) = 1$ and 1 is a divisor of 727, based on the theorem, there are integers x and y that satisfy the equation.

A particular solution is $x = 2;\ y = 145$

Another solution will be of the form x and y, so that

$$2x + 5y = 727$$

Since it is true that

$$2(1) + 5(145) = 727$$

Subtracting the equations gives:

$$2(x - 1) = 5(145 - y)$$

Then:

$x - 1$ is a multiple of 5: There are integers m, provided that: $x - 1 = 5m$

$145 - y$ is a multiple of 2. There are integers n, provided that: $145 - y = 2n$

Since

$$2(5m) = 5(2n)$$
$$m = n$$

And the solutions to the equation

$$2x + 5y = 727$$

are of the form: $x = 1 + 5m;\ y = 145 - 2m$, for any integer value of m .

If we want the values to correspond to points in the second quadrant, we must also require that $1 + 5m < 0$ and $145 - 2m > 0$, which is true for integers $m < -\frac{1}{5}$. That is, $m \in \{-1;\ -2; \ldots.\}$.

Arguments	Deductive logical justification based on identifying a particular solution and generating infinite solutions using divisibility properties.
Processes	Generalization, reification, algebraic symbolization and treatments with parameters

3.4 Meaning 4: Divisibility as a relationship that allows sufficient conditions to be determined to guarantee general results

It corresponds to mathematical practices that involve situations such as those described in S.4.1 and S.4.2.

S.4.1 Obtaining and justifying divisibility criteria. For example, what conditions must a 3-digit number meet to be a multiple of 7? What about a 4-digit number?

As an example, epistemic configuration is proposed for the described situation in Table 8.

Table 8

Epistemic configuration for situations requiring the proposal and justification of divisibility criteria.

Primary object/processes	Description
Situation	What conditions must a 3-digit number meet to be a multiple of 7? What about a 4-digit number?
Languages	Three-digit number abc $100a + 10b + c$ $(7)(14)a + 7b = 7k$
Definitions, concepts	Decomposition of a number in base 10. If N is a multiple of 7, then there is k in the integers, provided that $N = 7k$.
Properties	Properties of operations in $\mathbb{Z}$. Distributive property $a(b + c) = ab + ac$ If N and M are multiples of d, then $N + M$ and $N - M$ are multiples of d. Number a is a divisor of a. If a is a divisor of b and b is a divisor of c, then a is a divisor of c. If a is a divisor of b and c, then a is a divisor of $mb + nc$.
Procedures	We are looking for a relationship between a, b, and c that guarantees that abc is a multiple of 7. From the decomposition of number abc in base 10, we have that $100a + 10b + c$ $(98 + 2)a + (7 + 3)b + c$ $(\mathbf{7})(\mathbf{14})a + 2a + 7b + 3b + c$ $7k + 2a + 3b + c$ For that number to be a multiple of 7, it is required that $2a + 3b + c$ be a multiple of 7. $2a + 3b + c = 7k'$ Therefore, if $2a + 3b + c$ is a multiple of 7, then

	abc will also be a multiple of 7. If the number has 4 digits, it can be expressed as follows: $$1000a + 100b + 10c + d$$ $$(994 + 6)a + (98 + 2)b + (7 + 3)c + d$$ $$(\mathbf{7})(\mathbf{142})a + 6a + (\mathbf{7})(\mathbf{14})b + 2b + 7c + 3c + d$$ $$7k + 6a + 2b + 3c + d$$ For that number to be a multiple of 7, it is required that $6a + 2b + 3c + d$ be a multiple of 7. If $6a + 2b + 3c + d$ is a multiple of 7, then $abcd$ will also be a multiple of 7.
Arguments	Based on the decomposition in base 10 and the properties of operations in $\mathbb{Z}$ and divisibility, enough conditions are identified that guarantee the validity of a conditional proposition.
Processes	Generalization, reification (propose a general statement), algebraic symbolization and treatments with parameters.

S.4.2 Demonstrate properties derived from the relationship of divisibility. For example, prove that the product of two numbers is equal to the product of their least common multiple and greatest common divisor.

As an example, epistemic configuration is proposed for the described situation in Table 9.

Table 9

Epistemic configuration for situations requiring demonstration of divisibility properties

Primary object/processes	**Description**
Situation	Prove that the product of two numbers is equal to the product of their least common multiple and greatest common divisor.
Languages	$GCD(M; N), LCM(M; N)$ $N = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$
Definitions, concepts	The greatest common divisor of a and b is a divisor of a and b and is divisible by any other common divisor of a and b. The least common multiple of a and b is a

	multiple of a and b and is a divisor of any other common multiple of a and b. Prime factors; maximum and minimum of a set.
Properties	Properties of exponents. Given the set formed by two elements, $\{a; b\}$, where $a \leq b$, it is true that $máx\{a; b\} + mín\{a; b\} = a + b$.
Procedures	Given two numbers N and M, whose decomposition as a product of prime factors is $N = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$ $M = p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$ From the definitions of least common multiple and greatest common divisor, we have that: $LCM(N; M) = p_1^{máx\{r_1;s_1\}} p_2^{máx\{r_2;s_2\}} \dots p_k^{máx\{r_k;s_k\}}$ $GCD(N; M) = p_1^{mín\{r_1;s_1\}} p_2^{mín\{r_2;s_2\}} \dots p_k^{mín\{r_k;s_k\}}$ When multiplying, we have that: $LCM(N; M).GCD(N; M) = p_1^{máx\{r_1;s_1\}+mín\{r_1;s_1\}} p_2^{máx\{r_2;s_2\}+mín\{r_2;s_2\}} \dots p_k^{máx\{r_k;}$ Since the sets have only two elements, one of them will be the maximum and the other one the minimum. $LCM(N;\ M).GCD(N;\ M) = p_1^{r_1+s_1} p_2^{r_2+s_2} \dots p_k^{r_k+s_k}$ This result coincides with: $LCM(N;\ M).GCD(N;\ M) = N.M$
Arguments	Based on the properties of exponents, the decomposition of a number as a product of its prime factors, and the transitivity of the equality relation.
Processes	Algebraic symbolization, representation of families of numbers expressed as products of prime factors and syntactic activity.

3.5 Meaning 5: Divisibility as a relation in an integral domain (any commutative ring without zero divisors)

It corresponds to mathematical practices that involve situations such as those described in S.5.1.

S.5.1 Mathematical practices related to situations involving the prove that a set with certain operations is an integral domain or a unique factorization domain. For example, is it true that $\mathbb{Z}_3$ is an integrity domain?

As an example, epistemic configuration is proposed for the described situation in Table 10.

Table 10

Epistemic configuration of situations where it must be analyzed whether a set is a domain of integrity

Primary object/processes	Description
Situation	Is it true that $\mathbb{Z}_3$ is an integrity domain?
Languages	Module 3, remainder, zero divisors, commutative ring, etc.
Definitions, concepts	An integral domain is a commutative ring with identity and does not have zero divisors. $$\mathbb{Z}_3 = \{0; 1; 2\}$$ Addition and multiplication operations in $\mathbb{Z}_3$
Properties	Properties of addition and multiplication in $\mathbb{Z}$
Procedures	It can be verified that $\mathbb{Z}_3$ is a commutative ring with identity because: There is a neutral element: $0 + a = a$, for all $a \in \mathbb{Z}_3$. There is a multiplicative identity: $1.a = a$, for all $a \in \mathbb{Z}_3$ Multiplication is commutative: $a.b = b.a$ for all a y $b \in \mathbb{Z}_3$ Furthermore, $\mathbb{Z}_3$ has no zero divisors because when multiplying two elements other than zero, the result is not zero: $$1 \times 1 = 1; 1 \times 2 = 2; 2 \times 2 = 1$$
Arguments	Based on the definition, the conditions are verified. Logical deductive reasoning.
Processes	Generalization-particularization, algebraic symbolization, treatments and demonstration

ANALYSIS OF THE ALGEBRAIC REASONING FEATURES UNDERLYING THE SOLUTIONS TO DIVISIBILITY PROBLEMS

The levels of algebraization are assigned to the mathematical activity performed by the subject solving a mathematical problem or task and not to the tasks themselves, since these can be solved in different ways, involving different algebraic activities (Godino et al., 2015). Thus, it is possible to assess a task as being of one level or another, depending on the type of solution proposed (Castro et al., 2017). However, Llanes et al. (2022) point out that levels of algebraization can also be considered a theoretical and methodological tool with great potential to predict the mathematical activity associated with the development of EAR and used to analyze reference mathematical practices, such as the tasks proposed in textbooks, as this will allow the identification of which ones promote one level of algebraization or another.

Given that EAR levels proposal can be adapted to different mathematical objects (Gaita & Wilhelmi, 2019), this paper presents a proposal for EAR levels for divisibility. This analysis is carried out considering the features of the EAR (Table 1) that are revealed when analyzing the epistemic configurations associated with the different situations proposed for the partial meanings of divisibility, presented in the previous section. Each one mobilizes certain objects and processes of algebraic nature, with different degrees of structural complexity and generalization.

Complementarily, the classification of levels of understanding for divisibility presented by Espinoza and Pochulu (2020) is also considered to distinguish mathematical practices with more or less elaborate arguments. Based on the above, relationships are established between the levels of Elementary Algebraic Reasoning (EAR) and the partial meanings of divisibility (and their respective epistemic configurations), which are presented in Table 11.

Table 11

Partial meanings of divisibility and their algebraic features

Meaning	Problem situations	Objects and processes involved	Features of algebraic reasoning (EAR level)
M.1	S.1.1	Numerical representations (Languaje), divisor or multiple (Concept), arithmetic operations (Procedure), relationship between remainders and divisors (Proposition), performative practices: justification based on calculations (Argument).	Application of an algorithm between particular numbers, numerical language. (Level 0)
	S.1.2	In addition, S.1.2 requires identifying the form of the general term.	Identification of numerical patterns; use of generalizing language. (Level 1)
M.2	S.2.1	Natural language, numerical representations such as products and powers (Languaje), divisor, multiple, prime and composite numbers (Concept), successive divisions, prime factorization, identification of common factors (Procedure), divisibility properties (Proposition), argumentative practices: justification based on factorization and identification of common factors (Argument).	Application of the procedure for calculating all divisors of a number and systematic presentation of the results. (Level 1).
	S.2.2 S.2.3	In addition, in S.2.2 and S.2.3 there is a context that must be interpreted to determine what to do and what the result means.	Systematization and generalization of procedures for calculating the GCD or LCM for two or more numbers, use of

			intensives, referring to the context. (Level 1)
M.3	S.3.1	Symbolic and numerical representations and natural language (Languaje), linear equation with two unknowns, solution (Concept), identification of a solution, construction of other solutions explicitly or in terms of a parameter (Procedure), condition for the existence of a solution, properties of divisibility (Proposition), argumentative practices: logical justification based on the identification of a particular solution (Argument).	Symbolic representation, identification of structural relationships by adding even and odd numbers. (Level 2)
	S.3.2 S.3.3	Furthermore, in S.3.2 and S.3.3, the general solution is obtained from the transformation of symbolic expressions and is represented using variables.	Identification of structural relationships between multiples of 5 modulo 3, algebraic representation through two equations. (Level 3)
M.4	S.4.1	Symbolic representations (Languaje), Diophantine equations (Concept), symbolic representation of the structure underlying the condition and requirement of the problem, base 10 decomposition, transforming and obtaining equivalent expressions, both numerical and algebraic (Procedure), properties of divisibility, distributive property, modular congruences (Proposition), if then demonstration (Argument).	Formalization and use of variables (representation-meaning), identification of structural relationships between multiples, analytical calculations involving a parameter. (Level 4)
	S.4.2	In addition, S.4.2 requires factorization into prime factors, GCD, LCM (Concept) and properties of exponents (Proposition).	Representation using variables and parameters, distinguishing their role, justification of general

			rules analytical calculations involving several parameters. (Level 5)
M.5	S.5.1	Symbolic representations (Languaje), integrity domain as a commutative ring without zero divisors, zero divisors, a divides b, prime element, GCD, LCM (Concept), properties of addition and multiplication operations, conditions for the existence of GCD or LCM (Proposition), addition and multiplication operations (Procedure), demonstration based on the verification of conditions that define an integrity domain (Argument).	Formal generalization and reasoning about the algebraic structure of the integrity domain. (Level 6)

In this sense, solutions to divisibility situations that are attributed solely to an arithmetic procedure are associated with argumentative mathematical practices and show level 0 features in terms of EAR.

Meanwhile, solutions to divisibility situations that explicitly indicate the use of concepts and procedures are associated with argumentative practices and show level 1 or 2 features of EAR. Finally, solutions to divisibility situations that require formal deductive argumentative practices demand processes of generalization, reification, symbolization, and transformation of symbolized expressions are therefore associated with EAR level 3. On the other hand, solutions that require calculations involving one or more parameters correspond to level 4 or 5 features of the EAR, while reasoning about algebraic structures is a characteristic of level 6.

This organization of partial meanings of divisibility is not intended to be thorough; it seeks to offer an organized epistemological view of the situations/problems related to each one of the partial meanings, according to the proposal of mathematical objects and processes immersed in mathematical activity and the levels of algebraization of the onto-semiotic approach (Burgos & Godino, 2020). A certain gradual transition between arithmetic and algebraic reasoning can be glimpsed based on contexts of divisibility. In particular, situations that involve justifying criteria or solving Diophantine equations allow for the use of generalization and formalization processes that are more typical of algebraic thinking, while the early stages (search for divisors or GCD) can be considered pre-algebraic or an introduction to EAR.

CONCLUSIONS

Situations of divisibility have historically been approached from predominantly arithmetic perspectives, although several authors recognize their potential for algebraic reasoning due to the processes of generalization, symbolization, and argumentation that can be activated in the mathematical practices necessary for their resolution.

Although divisibility is introduced in primary education through the study of multiples, divisors, greatest common divisor, least common multiple, and divisibility criteria, it could be revisited in secondary school when solving Diophantine equations where the coefficients and solutions are integers. In teacher training, situations that require justifying and proposing divisibility criteria and demonstrating divisibility properties can also be considered, which will require the use of symbolic language, properties of integers, and the decimal number system, and articulating them following formal deductive reasoning. Thus, tasks on divisibility can be considered as an alternative arithmetic context to that of patterns and used in the design of training cycles that promote the development of EAR.

The solutions associated with the divisibility situations presented for each meaning have been analyzed using the Elementary Algebraic Reasoning (EAR) model, pointing out the algebraic objects and processes involved. This way, the features of the level of algebraization have been identified, which can be evidenced in the practices developed by students, by teachers, or expected by textbooks in the solution of a given problem.

Thus, with the identification of the meanings of divisibility and the association of features in terms of EAR, the hypothesis is confirmed: Divisibility problems allow the level of EAR possessed by subjects to be analyzed based on their practices. The potential of divisibility situations is recognized, evolving from those that require only the use of arithmetic procedures to those that demand recognition of the properties corresponding to the underlying algebraic structures.

Finally, the implementation of training processes for the development of EAR requires considering the specificity of the selected mathematical object, as in the case of proportionality or divisibility; this implies adapting the descriptions of the EAR levels, considering elements specific to the mathematical object, as it was done in section 5.

Experimental studies still need to be carried out so that, considering the types of situations presented and the appropriate modification of teaching variables, the development of mathematical practices can be promoted, requiring features associated with different levels of EAR. In particular, teaching sequences can be designed to be implemented during teacher training processes, both in initial training and in service.

AUTHORS' CONTRIBUTIONS STATEMENTS

All authors contributed to the conception and design of the study. The first draft of the manuscript was written by CGI, the first draft of the methodology was written by LPF and all authors commented on previous versions of the manuscript. The preparation of the material, formal analysis, and proposal of meanings and levels were carried out by CGI y LPF. Also, FUG validated the proposal of partial meanings of divisibility and he elaborated the descriptions of the mathematical objects presented for each of them. All authors actively participated in the discussion of the results, reviewed and approved the final version of the work.

DATA AVAILABILITY STATEMENT

Given the article's limited nature, the data supporting the findings and conclusions correspond to the review of texts and didactic and epistemological research where different uses of divisibility are implicitly revealed. Therefore, data sharing does not apply to this article, since it is based on publicly available literature.

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